Chapter 2

Numeration Systems

Notes

Contrary to what you may believe, there are many ways of expressing numbers. Some of these ways are cultural and historical. Others are different ways of thinking about what the digits of our conventional number system mean. For example, you probably think of 23 as meaning two tens and three ones. We think this way because we use a base ten system of counting. (How many fingers do you have?) But why not base five (using only five fingers)? What would 23 mean then? You are about to find out.

2.1 Ways of Expressing Values of Quantities

The need to quantify and express the values of quantities led humans to invent numeration systems. Throughout history, people have found ways to express values of quantities they measured in several ways. A variety of words and special symbols, called **numerals**, have been used to communicate number ideas. How one expresses numbers using these special symbols makes up a **numeration system**. Our Hindu-Arabic system uses ten **digits**: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Virtually all present-day societies use the Hindu-Arabic numeration system. With the help of decimal points, fraction bars, and marks like square root signs, these ten digits allow us to express almost any number and therefore the value of almost any quantity.

> ***Think About . . .***
> Why are numerals used so much? What are advantages of these special symbols over using just words to express numbers? What are some exceptions to representing numbers with digits?

Activity 1 You Mean People Didn't Always Count the Way We Do?

A glimpse of the richness of the history of numeration systems lies in looking at the variety of ways in which the number twelve has been expressed. In the different representations shown on the next page, see if you can deduce what each individual mark represents. Each representation expresses this many:

□□□□□□□□□□□□

Continue on the next page.

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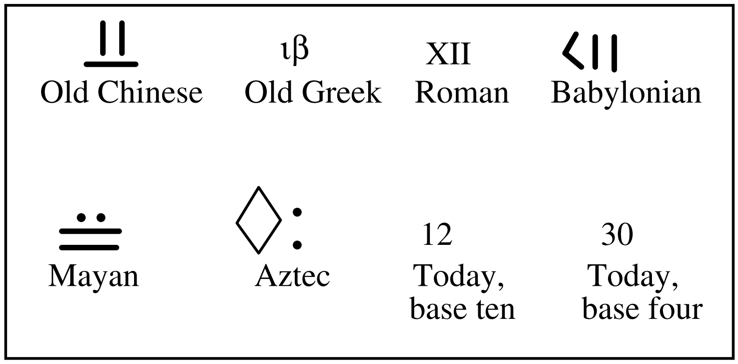

THINK ABOUT . . .

How would ten have been written in each of these earlier numeration systems?

Some ancient cultures did not need many number words. For example, they may have needed words only for "one," "two," and "many." When larger quantities were encountered, they could be expressed by some sort of matching with pebbles or sticks or parts of the body, but without the use of any distinct word or phrase for the number involved. For example, in a recently-discovered culture in Papua New Guinea, the same word "doro" was used for 2, 3, 4, 19, 20, and 21. But by pointing also to different parts of the hands, arms, and face when counting and saying "doro," these people could tell which number is intended by the word. This method of pointing allows the Papua New Guineans to express numbers up through 22 easily.[i] It was only when this culture came into contact with the outside world and began trading with other cultures that they needed to find ways of expressing larger numbers.

THINK ABOUT…

Why do you think we use ten digits in our number system? Would it make sense to use twenty? Why or why not?

Discussion 1 **Changing Complexity of Quantities Over Time**

What quantities, and therefore what number words, would you expect a caveman to have found useful? (Assume that the caveman had a sufficiently sophisticated language.) A person in a primitive agricultural society? A pioneer? An ordinary citizen living today? A person on Wall Street? An astronomer? A subatomic physicist?

TAKE-AWAY MESSAGE . . . Mathematical symbols have changed over the years, and they may change in the future. Symbols used for numbers depend upon our need to determine the value of the quantities with which we work. ♦

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Learning Exercises for Section 2.1

1. Based on what you have seen of the old counting systems such as Greek, Chinese, Roman, Babylonian, Mayan, and Aztec, which systems make the most sense to you? Explain.

2. Symbols for five and for ten often have had special prominence in geographically and chronologically remote systems. Why?

3. Numbers can be expressed in a fascinating variety of ways. Different languages, of course, use different words and different symbols to represent numbers. Some counting words are given below.

English	*Spanish*	*German*	*French*	*Japanese*	*Swahili*
zero	cero	null	zero	zero	sifuri
one	uno	eins	un	ichi	moja
two	dos	zwei	deux	ni	mbili
three	tres	drei	trois	san	tatu
four	cuatro	vier	quatre	shi	nne
five	cinco	fünf	cinq	go	tano
six	seis	sechs	six	roku	sita
seven	siete	sieben	sept	shichi	saba
eight	ocho	acht	huit	hachi	nane
nine	nueve	neun	neuf	kyu	tisa
ten	diez	zehn	dix	ju	kumi

Which two sets of these counting words most resemble one another?

Why do you think that is true? Do you know these numbers in yet another language?

4. Roman numerals have survived to a degree, as in motion picture film credits and on cornerstones. Here are the basic symbols: I = one, V = five, X = ten, L = fifty, C = one hundred, D = five hundred, and M = one thousand. For example, CLXI is 100 + 50 + 10 + 1 = 161. What numbers does each of these represent?

a. MMCXIII **b.** CLXXXV **c.** MDVII

5. How would each of the following be written in Roman numerals? For example, one thousand one hundred thirty would be MCXXX.

a. two thousand sixty-six **b.** seventy-eight **c.** six hundred five

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6. Other systems we have seen all involve addition of the values of the symbols. Roman numerals use a **subtractive** principle as well; when a symbol for a smaller value comes before the symbol for a larger value, the former value is subtracted from the latter. For example, IV means 5 – 1 = 4, or four; XC means 100 – 10 = 90; and CD = 500 – 100 = 400. Note that no symbol appears more than three times together, because with four symbols we would use this subtractive property. What number does each of these represent?

   a. CMIII **b.** XLIX **c.** CDIX

7. Even within the same language, there are often several words for a given number idea. For example, both "two shoes" and "a pair of shoes," refer to the same quantity. What are some other words for the idea of two-ness?

2.2 Place Value

What does each 2 in 22,222 mean? The different 2s represent different values because our **Hindu-Arabic numeration system** is a **place-value system**. This system depends upon the powers of ten to tell us the meaning of each digit. Once this system is understood, arithmetic operations are much easier to learn. Understanding of place value is a fundamental idea underlying elementary school mathematics.

But first, what does it mean to have a place-value system?

> In a **place-value** system, the value of a digit in a numeral is determined by its position in the numeral.

EXAMPLE 1

In 506.7, the 5 is in the hundreds place, so it represents five hundred. The 0 in 506.7 is in the tens place, so it represents zero tens, or just zero. The 6 is in the ones place, so it represents six ones, or six. And the 7 is in the tenths place, so it represents seven-tenths. The complete 506.7 symbol then represents the sum of those values: five hundred six and seven-tenths.

Notice that we do not say "five hundred, zero tens, six and seven-tenths," although we could. This is symptomatic of the relatively late appearance, historically, of a symbol for zero. The advantage of having a symbol to say that nothing is there is apparently a difficult idea, but the idea is vital to a place-value system. Would 506.7 mean the same number if we omitted the 0 to get 56.7? The 0 may have evolved from some type of round mark written in clay by the Babylonians to show that there are zero groups of a particular place value needed.

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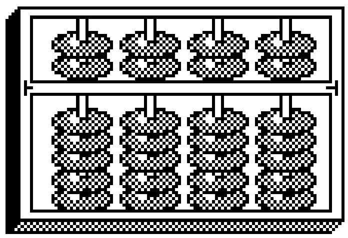
THINK ABOUT . . .

In the Hindu-Arabic place-value system, how many different places (positions) can you name and write numerically? (Don't forget places to the right of the decimal point.)

Before the use of numerals became widespread, much calculation was done with markers on lines for different place values. The lines could be on paper, or just drawn in sand, with small stones used as markers. (Our word "calculate" comes from the Latin word for "stones.") One device that no doubt was inspired by these methods of calculating is the **abacus**, which continues to be used in some parts of the world.



A Chinese Abacus

Notice that we have often used words to discuss the numbers instead of the usual numerals. The reason is that the symbol "12" is automatically associated with "twelve" in our minds because of our familiarity with the usual numeration system. We will find that the numeral "12" could mean five or six, however, in other systems! (If no base is indicated, assume the familiar base ten is intended.)

In our *base-ten* numeration system, the whole-number place values result from *groups of ten*—ten ones, ten tens, ten hundreds, etc. The digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 work fine until we have ten of something. But there is no single digit that means ten. When we have ten ones, we think of them as one group of ten, without any leftover ones, and we take advantage of place value to write "10," one ten and zero leftover ones. Similarly, two place values are sufficient through nine tens and nine ones, but when we have ten tens, we then use the next place value and write "100." It is like replacing ten pennies with one dime, and trading ten dimes for a dollar.

EXAMPLE 2

If I want to find the number of ten dollar bills I could get for $365, the answer is not just 6, it is 36.

If I want to know how many dollar bills I could get from $365, the answer is 365.

If I want to know the number of dimes I could get from $365, the answer is 3650.

But if I want to know how many tens are in 365, I could say either 36 or 36.5, depending on the context.

Continue on the next page.

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If I have 365 bars of soap and I want to know how many full boxes of 10 I could pack, the answer would be 36.

If I am buying 365 individual bars of soap priced at $6 per 10 bars, then I would have to pay 36.5 times $6.

With a good understanding of place value, the problems like those in Example 2 can be easily solved without undertaking long division or multiplication by 10 or powers of 10. Children who do not understand place value will often try to solve the problem of how many tens are in 365 by using long division to divide by 10, rather than observing that the answer is obvious from the number.

Discussion 2 Money and Place Value

Explain your answers to each of the following:

1. How many ten-dollar bills does the 6 in $657 represent? The 5?

2. How many tens are in 657?

3. How many one-hundred dollar bills can you get for $53,908?

4. How many one-hundreds are in 53,908?

5. How many pennies can you get for $347? For $34.70? For $3.47?

6. How many ones are in 347? In 34.70? In 3.47?

The decimal point indicates that we are beginning to break up the unit *one* into tenths, hundredths, thousandths, etc. But the number *one*, not the decimal point, is the focal point of this system. So 0.642 is 642 thousandths of *one*. Put another way, 0.6 is six tenths of *one*, while 6 is six *ones*, and 60 is six tens, or 60 *ones*. But just as 0.6 is six tenths of one, 6 is six tenths of 10, 60 is six tenths of one hundred, and so on up the line. Or starting with smaller numbers, 0.006 is six tenths of 0.01, while 0.06 is six tenths of 0.1. Likewise, 6000 is 60 hundreds, 600 is 60 tens, 60 is 60 ones, 6 is 60 tenths, 0.6 is 60 hundredths, 0.06 is 60 thousandths, and so on.[ii] While this at first might seem confusing, it becomes less so with practice and thought.

Take-Away Message . . . Our base ten place-value numeration system is adequate for expressing all whole numbers and many decimal numbers. The value of each digit in a numeral is determined by the position of the digit in the numeral. Digits in different places have different values. Finally, the number 1, not the decimal point, serves as the focal point of decimal numbers. ♦

Learning Exercises for Section 2.2

1. **a.** How many tens are in 357? How many whole tens?

b. How many hundreds are in 4362? How many whole hundreds?

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c. How many tens are in 4362? How many whole tens?

d. How many thousands are in 456,654? How many whole thousands?

e. How many hundreds are in 456,654? How many whole hundreds?

f. How many tens are in 456,654? How many whole tens?

g. How many tenths are in 23.47? How many whole tenths?

h. How many thousandths are in 23.47? How many whole thousandths?

i. How many ones are in 23.47? How many whole ones?

j. How many hundredths are in 23.47? How many whole hundredths?

k. How many tenths are in 2347? How many whole tenths?

l. How many tenths are in 234.7? How many whole tenths?

2. In 123.456, the hundre*d*s place is in the third place to the left of the decimal point; is the hundred*th*s place in the third place to the right of the decimal point? In a long numeral like 333331.333333, what separates the number into two parts that match in the way hundreds and hundredths do?

3. a. Is the statement "For a set of whole numbers, the longest numeral will belong to the largest number" true or false? Why?

b. Is the statement "For a set of decimals, the longest numeral will belong to the largest number" true or false? Why?

4. Pronounce 3200 in two different ways. Do the two pronunciations have the same value?

5. Write in words the way you would pronounce each:

a. 407.053 **b.** 30.04 **c.** 0.34 **d.** 200.067 **e.** 0.276

6. Each of the following represents work of students who did not understand place value. Find the errors made by these students, and explain their reasoning.

a.
$$\begin{array}{r} 15 \\ +\underline{95} \\ 1010 \end{array}$$

b.
$$\begin{array}{r} 55 \\ +\underline{48} \\ 913 \end{array}$$

c.
$$\begin{array}{r} 7 \\ 4\,^{1}\!\not{8} \\ \underline{-2\;6} \\ 1\;1 \end{array}$$

d.
$$\begin{array}{r} 36 \\ 7\overline{)43} \\ \underline{42} \\ 22 \\ \underline{21} \end{array}$$

e.
$$\begin{array}{r} 36 \\ \underline{\times\;\;8} \\ 2448 \end{array}$$

7. In base ten, 1635 is exactly ________ *ones,* is exactly _________ *tens,* is exactly _________ *hundreds,* is exactly _________ *thousands;* it is also exactly __________ *tenths,* or exactly _________ *hundredths.*

8. In base ten, 73.5 is exactly ________ *ones,* is exactly _________ *tens,* is exactly _________ *hundreds,* is exactly _________ *thousands;* it is also exactly __________ *tenths,* or exactly _________ *hundredths.*

9. Do you change the value of a whole number by placing zeros to the right of the number? To the left of the number?

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2.3 Bases Other Than Ten

Too often children learn to operate on numbers without having a deep understanding of place value, the lack of which leads them to make many computational errors. The purpose of this section is to provide experiences with base numeration systems other than ten so you understand the underlying structure of the base ten system of numeration. You are not expected to become fluent in a base other than ten. Rather, you should be able to calculate in different bases to the extent that is needed to understand the role of place value in calculations.

THINK ABOUT . . .

We use a base ten system of counting because we have ten fingers. Other cultures have used other bases. For example, some Eskimos were found to count using base five. Why would that be? What other bases might have been used for counting?

Cartoon characters often have three fingers and a thumb on each hand, a total of eight fingers (counting thumbs) instead of ten. Suppose that we live in this cartoon land and instead of having ten digits in our counting system (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) we have only eight digits (0, 1, 2, 3, 4, 5, 6, 7). Using this new counting system we write the number eight as 10_{eight}, meaning 1 group of eight and 0 ones. Thus, we would write as we count in base eight:

1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 20, 21, ...

We read this list of numbers as: one, two, three, . . . , one-zero, one-one, one-two, . . . , two-zero, . . .

Activity 2 Place Value in Cartoon Land

1. Show the value of each place in base eight by completing this pattern:

. . .	___	___	___	___	___	___
	8^5	8^4	8^3	8^2	8^1	8^0
	?	**?**	**?**	sixty-fours	eights	ones

2. What would follow 77 in base eight?

3. What would each digit indicate in the numeral 743 in base eight?

Notice that the base-eight numeration system has eight digits, 0–7. Writing 6072 in base eight would require the use of the first four places to the left of the decimal point and represents 2 ones (8^0), 7 eights (8^1), 0 sets of eight squared (8^2), and 6 sets of eight cubed (8^3). The digits 6, 0, 7, and 2 would be placed in the Activity pattern in the four places to the left of the decimal point. We call this number "six zero seven two, base eight" and write it as 6072_{eight}.

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THINK ABOUT . . .
If you had 602_{eight} chairs in an auditorium, how many chairs would you have, written in base ten?

Discussion 3 Place Value in Base Three

What are the place values in a base three system? What are the digits, and how many do we need? (Rather than invent new symbols for digits, let's use whichever of the standard symbols we need.) Study the chart below. What should be in place of the question marks?

Items	Name in base ten	Name in base three	Base three symbol
	zero	zero	0_{three}
□	one	one	1_{three}
□ □	two	two	2_{three}
□ □ □	three	???	???

Naming *three* in base three is a key step in understanding base three. Since there are three single boxes above, they will be grouped to make *one group of three*, and the base three symbol is 10! Notice that in base three "10" does not symbolize *ten* as we think about ten. In base three, "10" means "one group of three and zero left over." Since it does not mean ten, we should not pronounce the numeral as "ten." The recommended pronunciation is "one zero, base three," saying just the name for each digit and for the base. Notice how this chart differs from the one above.

Items	Name in base ten	Name in base three	Base three symbol
	zero	zero	0_{three}
□	one	one	1_{three}
□ □	two	two	2_{three}
□□□	three	one-zero	10_{three}
□□□ □	four	one-one	11_{three}

If there are four boxes, as in the last line of this table, we can make one group of three, and then there will be one left-over box, so in base three, four is written "11". Because we have the strong link between the marks "11" and eleven from all of our base ten experience, the notation 11_{three} is often used for clarity to show that the

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symbols should be interpreted in base three. Recall that 11_{three} should be pronounced "one one, base three," and not as "eleven."

Activity 3 Count in Base Three and in Base Four

Continue to draw more boxes and to write base three symbols. What do you write for five boxes? (Now you see why the symbol 12 might mean five.) Six? Seven? Eight? And, at another dramatic point, nine? Did you write "100_{three}" for nine? What would 1000_{three} mean?

Check your counting skills by following along with counting in base four: 1, 2, 3, 10, 11, 12, 13, 20, 21, 22, 23, 30, 31, 32, 33, 100, 101, 102, 103, 110, 111, 112, 113, 120, 121, 122, 123, 130, 131, 132, 133, 200 . . .
What does 1000_{four} mean?

Discussion 4 Working with Different Bases

1. What are the place values in base five? What digits are needed? How would thirty-eight (in base ten) be expressed in base five? Record the first fifteen counting numbers in base five: 1, 2, . . .
2. What are the place values in a base *b* place-value system? What digits are needed?
3. What are the place values in a base-two place-value system? How would eighteen (in base ten) be written in base two? The inner workings of computers use base two; do you see any reason for this fact?
4. Perhaps surprisingly, there is a Duodecimal Society, which promotes the adoption of a base twelve numeration system. What are the place values in a base twelve system? What new digits would have to be invented?

With several numeration systems possible, there can be many "translations" among the symbols. For example, given a base ten numeral (or the usual word), find the base six (or four or twelve) numeral for the same number, and vice versa, given a numeral in some other base, find its base ten numeral (or the usual word). In each case, the key is knowing, and probably writing down, the place values in the unfamiliar system. (Recall that any nonzero number to the 0 power is 1. Example: $5^0 = 1$.)

EXAMPLE 3

Changing from a non-ten base to base ten: What does 2103_{four} represent in base ten?

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SOLUTION

1. 2103_{four} has four digits. The first four place values in base four are written here, and the given digits put in their places:

2	1	0	3
of four sixteens, or sixty-four, or 4^3	of four fours, or sixteen, or 4^2	of four ones, or four, or 4^1	ones, or 4^0

2. What does the 2 tell us? The 2 stands for two of 4^3 which is $2 \times 64 = 128$ in base ten.
3. What does the 1 tell us? The 1 stands for one 4^2 which is $1 \times 16 = 16$ in base ten.
4. What does the 0 tell us? The 0 stands for zero of 4^1 which is $0 \times 4 = 0$ in base ten.
5. What does the 3 tell us? The 3 indicates 4^0 is used three times, $3 \times 1 = 3$ in base ten.
6. Thus $2103_{four} = (128 + 16 + 0 + 3)_{ten} = 147_{ten}$ that is, $2103_{four} = 147_{ten}$.

EXAMPLE 4

Suppose instead we want to change a number written in base ten, say 236, to a number written in another base, say base five. We know that the places in base five are the following:

. . . ______	______	______	______
one-hundred-twenty-fives (5^3)	twenty-fives (5^2)	fives (5^1)	ones (5^0)

SOLUTION

(You may find these steps easier to follow by dropping the ten subscript for now, for numbers in base ten.)

1. Look for the highest power of 5 in the base ten number; here it is 5^3 because 5^4 is 625_{ten} and 625_{ten} is larger than 236_{ten}. Are there any 5^3s in 236_{ten}? Yes, just one 5^3 because $5^3 = 125$, and there is only one 125 in 236. Place a 1 in the first place above to indicate one 5^3. Now you have "used up" 125, so subtract: $236_{ten} - 125_{ten} = 111_{ten}$.
2. The next place value of five is 5^2. Are there any twenty-fives in 111_{ten}? There are 4, so place a 4 above 5^2. Now four twenty-fives, or 100, have been "used," and $111_{ten} - 100_{ten} = 11_{ten}$.

Continue on the next page.

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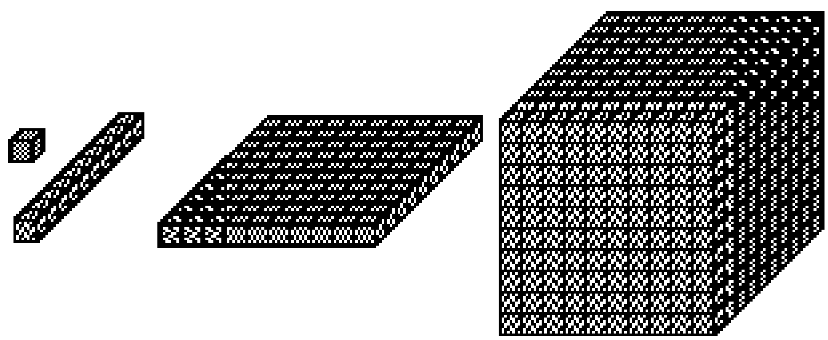
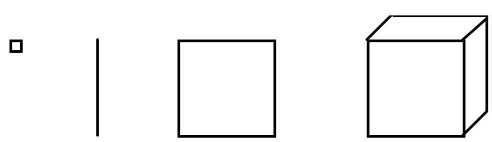
3. The next place value is 5^1 which is 5. How many fives are in 11_{ten}? It has two fives, so place 2 above 5^1. There is 1 one left, so place a 1 above 5^0. Thus $236_{ten} = 1421_{five}$.

Working with different bases can be easier when one can physically move pieces that represent different values in a base system. Often, after doing physical manipulation, one can mentally picture the manipulation and work without physical objects. Multi-base blocks are manipulatives that have proven to be extremely useful in coming to understand any base system, but primarily base ten in elementary school. Multibase blocks are wooden or plastic blocks that can be used to demonstrate operations in different bases. For base ten, a centimeter cube can be used to represent a unit or one; a long block one centimeter by one centimeter by ten centimeters (often marked in ones) would then represent ten; ten longs together form a flat that is one cm by ten cm by ten cm and that represents one hundred, ten flats form a ten cm by ten cm by ten cm cube that represents thousands. If the long is used for the unit, then the small cube would represent one-tenth, the flat would represent ten, and so on. The multi-base blocks can be used to strengthen place value understanding.



If the multibase blocks are not available, then they can easily be sketched as shown below:



The materials are often called "small cube, long, flat, big cube." Any size of the multibase blocks can be used to represent one unit. Familiarize yourself with the multibase blocks by doing this section's Learning Exercises and making up more problems until you feel you are familiar with the blocks and their relationships.

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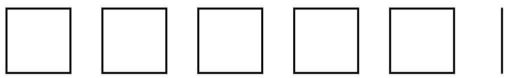
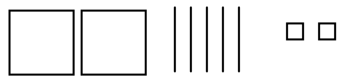
EXAMPLE 5

The sketch below represents numbers with bases larger than five because there are five flats. If the little cube represents the unit one, the number here is 520 for any base larger than five. If the long represents one, then the number represented here is 52 in any base larger than five. If the flat represents one, then the number represented here is 5.2 in any base larger than five.

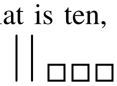


Discussion 5 Representing Numbers with Multibase Drawings

1. Here is a representation of a number: 

   Which bases could use this representation if it is in the final form, with no more "trades" possible? Why? What are some possible numbers that can be represented by this drawing?

2. In base eight, how many small cubes are in a long? How many in a flat? How many in a large cube? How many longs in a flat? How many flats in a large cube? Answer the same questions for base ten; for base two.

One can also represent decimal numbers with base ten blocks or drawings. You must first decide which block represents the unit. If the unit is the long, then the small block is one-tenth, the flat is ten, and the large block is 100. Thus 2.3 in base ten could be represented as: 

Activity 4 Representing Numbers with Multibase Blocks

For these problems, use your cutout blocks (from the appendix) or use drawings such as shown above. Note that the drawings do not show the markings of the base that appears in the picture of the blocks, and thus do not clearly indicate the base in the way that multibase blocks do.

1. Represent 2.3 in base ten using the long as one unit. Represent 2.3 using another size of block as the unit. Compare your representation with a neighbor.
2. Use the base five blocks to represent 2.41_{five} in two different ways. Be sure to indicate which piece represents the unit in each case.

TAKE-AWAY MESSAGE . . . We could just as easily have based our number system on something other than ten, but ten is a natural number to use because we have ten fingers. By working in bases other than ten, you have probably gained a new perspective on the structure and complexity of our place value system, particularly the importance of the value of each place. This understanding underlies all of the procedures we use in

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calculating with numbers in base ten. As teachers, you will need this knowledge to help students understand computational procedures. ♦

Learning Exercises for Section 2.3

1. If you have access to the internet, go to http://nlvm.usu.edu/en/nav/ and find Virtual Library, then Numbers and Operations, then 3-5, then to Base Blocks. You cannot choose numbers to represent, but you can set the base and you can set the number of decimal points. Practice doing this with the following:

a. whole numbers in base ten,

b. decimal numbers in base ten,

c. whole numbers in base five,

d. "basimal" numbers in base five.

2. Write ten (this many: 🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎) in each given system.

a. base four **b.** base five **c.** base eight

3. Write each of these.

a. four in base four **b.** eight in base eight

c. twenty in base twenty **d.** b in base b

e. b^2 in base b **f.** $b^3 + b^2$ in base b

g. 29_{ten} in base three **h.** 115_{ten} in base five

i. 69_{ten} in base two **j.** 1728_{ten} in base twelve

4. Write the numerals for counting in base two, from one through twenty.

5. How do you know that there is an error in each statement?

a. ten = 24_{three}

b. fifty-six = 107_{seven}

c. thirteen and three-fourths = 25.3_{four}

6. Write each of these as a base ten numeral with the usual base ten words. For example, $111_{two} = (1 \times 2^2) + (1 \times 2) + (1 \times 1) = 7_{ten}$ and $31.2_{four} = (3 \times 4) + (1 \times 1) + \frac{2}{4} = 12 + 1 + \frac{5}{10} = 13.5$, or thirteen and five-tenths.

a. 37_{twelve} **b.** 37_{nine} **c.** 207.0024_{ten}

d. 1000_{two} **e.** $1{,}000{,}000_{two}$ **f.** 221.2_{three}

7. For a given number, which base—two or twelve—will usually have a numeral with more digits? What are the exceptions?

8. In what bases would 4025_b be a legitimate numeral?

9. Compare these pairs of numbers by placing $<$ or $>$ or $=$ in each box.

a. 34_{five} ☐ 34_{six} **b.** 4_{five} ☐ 4_{six} **c.** 43_{five} ☐ 25_{six}

d. 100_{five} ☐ 18_{nine} **e.** 111_{two} ☐ 7_{ten} **f.** 23_{six} ☐ 23_{five}

Notes

10. On one of your space voyages, you uncover an alien document in which some "one, two, . . ." counting is done: obi, fin, mus, obi na, obi obi, obi fin, obi mus. What base does this alien civilization apparently use? Continue counting through twenty in that system.

11. Hints of the influence of other bases remain in some languages. What base could have led to each of these?

a. French for eighty is *quatre-vingt.*

b. The Gettysburg Address, "Four score and seven years ago . . ."

c. A gross is a dozen dozen.

d. A minute has 60 seconds, and an hour has 60 minutes.

12. What does 34.2_{five} mean? What is this number written in base ten?

13. In each number, write the "basimal" place values and then the usual base ten fraction or mixed number.

Example: $10.111_{two} = (2 + 0 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8})_{ten} = 2\frac{7}{8}$ (Recall: $4 = 2^2$ and $8 = 2^3$.)

a. 21.23_{four} **b.** 34.3_{twelve}

14. Write each of these in "basimal" notation.

Example: three-fourths in base ten is *what* in base two?

$$\left(\frac{3}{4}\right)_{ten} = \left(\frac{1}{2} + \frac{1}{4}\right)_{ten} = 0.11_{two}$$

a. one-fourth, in base twelve

b. three-fourths, in base twelve

c. one-fourth, in base eight

15. Give the base ten numeral for each given number.

a. 101010_{two} **b.** 912_{twelve} **c.** 425_{six}

d. 41.5_{eight} **e.** 1341_{five}

16. Write this many ♦♦♦♦♦♦♦♦♦♦♦♦ in each given base. (Note that there are 12_{ten} diamonds.)

a. nine **b.** eight **c.** seven **d.** six **e.** five **f.** four **g.** three **h.** two

17. Write 100_{ten} in each given base.

a. seven **b.** five **c.** eleven **d.** two **e.** thirty-one

18. Complete with the proper digits.

a. 57_{ten} = ______ five **b.** 86_{nine} = ______ ten

c. 312_{four} = ______ ten **d.** 237_{ten} = ______ eight

e. 2101_{three} = ______ ten **f.** 0.111_{two} = ______ ten

19. Represent 34 in base ten, with the small block as the unit; with the long as the unit.

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20. **a.** Represent 234_{five} with the small cube as the unit. (Notice that 234 does not mean two-hundred thirty-four here.)

b. Represent 234_{six} with the small cube as the unit.

(If you have only base ten blocks available, then sketch drawings for these exercises.)

21. In base six, 5413 is _________ *ones,* is _________ *sixes,* is _________ *six*^{2}*s;* is __________ *six*^{3}*s.*

22. Represent 2.34 in base ten with the flat as the unit.

23. Decide on a representation with base ten blocks for each number.

a. 3542 **b.** 0.741 **c.** 11.11

24. Represent 5.4 and 5.21 with base ten blocks, using the same block as the unit. (What will you use to represent one?) Many school children say that 5.21 is larger than 5.4 because 21 is larger than 4. How would you try to correct this error using base ten blocks?

25. Someone said, "A number can be written in many ways." Explain that statement.

2.4 Operations in Different Bases

Just as we can add, subtract, multiply, and divide in base ten, so can we perform these arithmetic operations in other bases. The standard algorithm for addition, depicted first below, is commonly used and is probably known to all of you. The expanded algorithms make the processes easier to understand. Once it is well understood, an expanded algorithm is easily adapted to become the standard algorithm. Not all standard algorithms in this country are used in other countries, so the word "standard" is a relative one.

In base ten we could add 256 and 475 in these two ways, as shown here. (There are other ways, of course.) The first way is called an *expanded* algorithm, and the second, called the standard algorithm, is probably the one you were taught.

```
  256                                11
+ 475                               256
   11  (thinking 6 + 5)           + 475
  120  (thinking 50 + 70)           731
  600  (thinking 200 + 400)
  731
```

The expanded algorithm is now being taught in some schools as a preparation for the standard algorithm. Note how place value is attended to in the expanded algorithm: add the ones 6 + 5, then add the tens 50 + 70, then add the hundreds, 200 + 400, then add the resulting sums, 11 + 120 + 500. In the standard algorithm, each "column" is

Notes

treated the same: 6 + 5 in the column on the right, 5 + 7 +1 in the middle column, and 2 + 4 + 1 in the column to the left. Although the standard algorithm leads to the correct answer, students frequently do not know why each step is taken. But when the expanded algorithm is understood, it can be condensed into the standard algorithm as shown above.

We can also use either method for adding in other bases, but the expanded algorithm is sometimes easier to follow until adding in another base is well understood.

EXAMPLE 6

Here is an example using both the standard and expanded algorithms to add the same two numbers in base ten and base eight. Make sure you can understand each way in each given base.

$$\begin{array}{r} {}^{1} \\ 351_{ten} \\ \underline{+\ 250}_{ten} \\ 601_{ten} \end{array} \qquad \begin{array}{rl} 351_{ten} & \\ \underline{+250}_{ten} & \\ 1_{ten} & \text{thinking } (1 + 0) \\ 100_{ten} & \text{thinking } (50 + 50) \\ \underline{500}_{ten} & \text{thinking } (300 + 200) \\ 601_{ten} & \text{thinking } (1 + 100 + 500) \end{array}$$

$$\begin{array}{r} {}^{1} \\ 351_{eight} \\ \underline{+\ 250}_{eight} \\ 621_{eight} \end{array} \qquad \begin{array}{rl} 351_{eight} & \\ \underline{+\ 250}_{eight} & \\ 1_{eight} & \text{thinking } (1 + 0)_{eight} \\ 120_{eight} & \text{thinking } (50 + 50)_{eight} \\ \underline{500}_{eight} & \text{thinking } (300 + 200)_{eight} \\ 621_{eight} & \text{thinking } (1 + 20 + 100 + 500)_{eight} \end{array}$$

Activity 5 Adding in Base Four

Add these two numbers in base four in both expanded and standard algorithms: 311_{four} and 231_{four}. (Drawings of base four pieces may be helpful.)

If we can add in different bases, we should be able to subtract in different bases. Here is an example of how to do this.

EXAMPLE 7

Find $321_{five} - 132_{five}$.

One way to think about this problem is to regroup in base five just as we do in base ten, then use the standard way of subtracting in base ten.

Continue on the next page.

Notes

SOLUTION:

$$\begin{array}{r} 321 \\ \underline{-\ 132} \end{array}$$

Step 1: We cannot remove 2 ones from 1 one, so we need to take one of the fives from 321_{five} and trade it for five ones:

$$321_{five} \rightarrow 300_{five} + 20_{five} + 1_{five} \rightarrow 300_{five} + 10_{five} + 11_{five}$$

Step 2: We can now take 2 ones from 11 ones (in base five) leaving 4 ones. (Notice how 321 has changed with 3 five squared, then 1 five, then 11 ones, from Step 1.)

$$\begin{array}{rrl} & 1 & \\ 3 & \not{2} & {}^{1}1_{five} \\ \underline{-1} & \underline{3} & \underline{2}_{five} \\ & & 4_{five} \end{array}$$

means 3 (five squared) + 1 five + 11 ones as in Step 1.

Step 3: In the fives place: We cannot subtract 3 fives from 1 five, so we must change one five squared to five sets of five. This, together with the one five already in place, gives us 11 fives (or six fives).

That is: $300_{five} + 10_{five} \rightarrow 200_{five} + 110_{five}$, so

$$\begin{array}{rrl} 2 & 11 & \\ \not{3} & \not{2} & {}^{1}1_{five} \\ \underline{1} & \underline{3} & \underline{2}_{five} \\ 1 & 3 & 4_{five} \end{array}$$

means 11 fives, not 11 ones, so the 11 stands for 110 and 11 fives minus 3 fives is 3 fives, or 110 – 30 is 30)

We now have 2 (five squared) from which 1 (five squared) is subtracted, leaving 1 (five squared). The answer is 1 five-squared plus 3 fives plus 4 ones which is 134_{five}.

Activity 6 Subtracting in Base Four

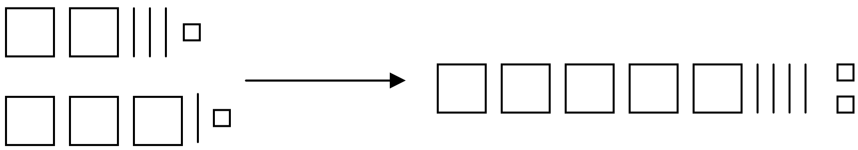
Subtract 231_{four} from 311_{four} in base four.

Subtracting in base four is similar to adding in base four. However, for both operations we can use base materials to help visualize adding and subtracting in other bases. We will do that next. You can cut out and use materials from an appendix on bases. As you use the base materials, notice how they support the symbolic work you did earlier in this section.

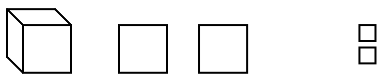
EXAMPLE 8

Suppose we want to add 231_{four} and 311_{four} using base four blocks, using the small block as the unit. We could first express the problem as

Notes



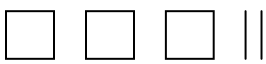
We have too many longs (in base four), so trade four longs for a flat. Now we have too many flats (each representing four squared). Trade four flats for a large cube (which represents four cubed).



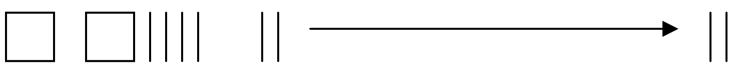
Represents the answer, which is 1202_{four}
The blocks represent one four cubed, two four squared, and two ones.

EXAMPLE 9

Suppose we want to subtract 23_{four} from 32_{four}. This time let us use the *long* as the unit. 32_{four} is represented:



I cannot remove 3 longs (ones) until I change a flat to four longs (which means change one four into four ones).



Remove two flats; three longs

To take away 23_{four} we must remove three longs (three ones), and 2 flats (2 fours), and we are left with 3_{four} as the difference.

THINK ABOUT . . .

If we had used the small block as the unit in the above subtraction example, would the numerical answer be different? Try it.

Activity 7 **Subtracting in Base Four**

Once again, subtract 231_{four} from 311_{four} in base four, this time using drawings.

We can also multiply and divide in different bases. However, the intent here is to introduce you to different bases so that you have a better understanding of our own base ten system, and that you understand why children need time to learn to operate in base ten. Thus there are no examples or exercises provided here for multiplication and division in different bases, although it is certainly possible to carry out these operations.

Notes

TAKE-AWAY MESSAGE . . . Arithmetic operations in other bases are undertaken in the same way as in base ten. However, because we have less familiarity with other bases, arithmetic operations in those bases take us longer than operations in base ten. For children not yet entirely familiar with base ten, time needed to complete arithmetic operations takes longer than it does for us. ♦

Learning Exercises for Section 2.4

1. Add 1111_{three} and 2102_{three} without drawings and then with drawings in the ways illustrated above. Which way did you find it easier?

2. Do these exercises in the designated bases, using the cardboard cutouts in an appendix, or with drawings.

a. $341_{\text{five}} + 220_{\text{five}}$ **b.** $101_{\text{two}} + 110_{\text{two}}$ **c.** $321_{\text{four}} - 123_{\text{four}}$ **d.** $296_{\text{ten}} - 28_{\text{ten}}$

3. Go to http://nlvm.usu.edu/en/nav/ on the internet and find Virtual Library, then Numbers and Operations, then 3-5. Go to Base Blocks Decimals. You cannot choose numbers to add and subtract, but you can set the base and you can set the number of decimal points. Do the following:

a. Practice adding and subtracting numbers in base ten using whole numbers.

b. Practice adding and subtracting numbers using one decimal place.

c. Practice adding and subtracting numbers in base four using whole numbers.

d. Practice adding and subtracting numbers in base four using one "decimal" place.

4. Add the following in the appropriate bases, without blocks unless you need them.

a. $2431_{\text{five}} + 223_{\text{five}}$ **b.** $351_{\text{nine}} + 250_{\text{nine}}$ **c.** $643_{\text{seven}} + 134_{\text{seven}}$ **d.** $99_{\text{eleven}} + 88_{\text{eleven}}$

5. Subtract in different bases, without blocks unless you need them.

a. $351_{\text{nine}} - 250_{\text{nine}}$ **b.** $643_{\text{seven}} - 134_{\text{seven}}$ **c.** $2431_{\text{five}} - 223_{\text{five}}$ **d.** $772_{\text{eleven}} - 249_{\text{eleven}}$

6. Do you think multiplying and dividing in different bases would be difficult? Why or why not?

7. Use the cut-outs from the appendix for the different bases to act out the following. As you act each out, record what would take place in the corresponding numerical work.

a. 232_{four}, 13_{four}, 113_{four} **b.** $232_{\text{five}} + 13_{\text{five}} + 113_{\text{five}}$ **c.** $232_{\text{eight}} + 13_{\text{eight}} + 113_{\text{eight}}$ **d.** $101_{\text{two}} + 11_{\text{two}} + 111_{\text{two}}$

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8. Use the cut-outs from an appendix for the different bases to act out the following. As you act each out, record what would take place in the corresponding numerical work.

a. 200_{four} $\underline{-13}_{\text{four}}$ **b.** 200_{five} $\underline{-13}_{\text{five}}$ **c.** 200_{eight} $\underline{-13}_{\text{eight}}$ **d.** 100_{two} $\underline{-11}_{\text{two}}$

9. Describe how cut-outs for base six would look. For base twelve.

2.5 Issues for Learning: Understanding Place Value

The notion that ten ones and one ten give the same number is vital to understanding the usual numeration system, as are the later rethinking of ten tens as one hundred, ten hundreds as one thousand, etc. Understanding place value is considered to be foundational to elementary school mathematics.

But base ten for children might be as mysterious as base *b* may have been for you. (Admittedly, your extensive experience with base ten also gets in the way!) By working with other bases, you have had the opportunity to explore what it means to have a place-value system where each digit has a particular meaning, and thus come to a better understanding of our base ten system of writing numbers and calculating with numbers.

One activity-centered primary program incorporates many activities involving grouping by twos, by threes, and so on, even before extensive work with base ten groupings, to accustom the children to counting not just one object at a time, but groups each made up of several objects. Ungrouping needs to be included also. That is, 132 could be regarded as one one-hundred, three tens, and 2 ones. Or, it could be regarded as one one-hundred and 32 ones. Here, the 3 tens are "unbundled" to make 30 ones. Regarding a group made up of several objects as one thing is a major step that needs instructional attention.

The manner in which we vocalize numbers can sometimes cause problems for students. For example, some young U.S. children will write 81 for eighteen, whereas scarcely any Hispanic children (diez y ocho = eighteen) or Japanese children (ju hachi = eighteen) do so. (Some wishfully think we should say "onetyeight" for eighteen in English.) What other numbers can cause the same sort of problem that eighteen does?

Place value instruction in schools is often superficial and limited to studying only the placement of digits. Thus, children are taught that the 7 in 7200 is in the thousands place, the 2 is in the hundreds place, a 0 is in the tens place, and a 0 is in the ones place. But when asked how many hundred dollar bills could be obtained from a bank account with $7200 in it, or how many boxes of ten golf balls could be packed from a container with 7200 balls, children almost always do long division, dividing by 100 or by 10. They do not read the number as 7200 ones, or 720 tens, or 72 hundreds, and certainly not as 7.2 thousands. But why not? These are all names for the same

Notes

number, and the ability to rename in this way provides a great deal of flexibility and insight when working with the number. (It is interesting that we later expect students to understand newspaper figures such as $3.2 billion. What does .2 billion mean here?)

Over the years many different methods have been used to teach place value. An abacus with nine beads on each string is one type of device used to represent place value. The Base Ten Blocks pictured in Section 2.3 have been extensively used to introduce place value and operations on whole numbers and decimal numbers. One problem with these representations, however, is that students do not always make the connections between what is shown with the manipulative devices and what they write on paper.

Our place value system of numeration extends to numbers less than 1 also. The naming of decimal numbers needs special attention. The place value name for 0.642 is six hundred forty-two thousandths. Compare this to reading 642, where we simply say six hundred forty-two, not 642 ones. This is a source of confusion that is compounded by the use of the ten*ths* or hundred*ths* with decimal numbers, the use of te*n* or hundre*d* with whole numbers, and the additional digits in the whole number with a similar name. The number 0.642 is read 642 thousand*th*s, meaning 642 thousand*th*s of one, while 642,000 is read 642 thousand, meaning 642 thousand ones. That tens and ten*th*s, hundreds and hundred*th*s, etc., sound so much alike no doubt causes some children to lose sense-making when it comes to decimals. Some teachers resort to a digit-by-digit pronunciation—"two point one five" for 2.15—but that removes any sense for the number; it just describes the numeral. Plan to give an artificial emphasis to the -th sound when you are discussing decimals with children. (You can also say "decimal numeral two and three-tenths" and "mixed numeral two and three-tenths" to distinguish 2.3 and $2\frac{3}{10}$.)

To compare 0.45 and 0.6, students are often told to "add a zero so the numbers are the same size." (Try figuring out what this might mean to a student who does not understand decimal numbers in the first place!) The strategy works, in the sense that the student can then (usually) choose the larger number, but since it requires no knowledge of the size of the decimal numbers, it does not develop understanding of number size. Instead of annexing zeros, couldn't we expect students to recognize that six-tenths is more than forty-five hundredths because 45 hundredths has only 4 tenths and what is left is less than another tenth? But for students to do this naturally, they must have been provided with numerous opportunities to explore—and think about—place value. Comparing and operating on decimals, if presented in a non-rule oriented fashion, can provide these opportunities. If teachers postpone work with operations on decimals until students conceptually understand these numbers, students will be much more successful than if teachers attempt to teach computation too early. Some researchers[iii] have shown that once students have learned rote rules for calculating with decimals, it is extremely difficult for them to relearn how to calculate with decimals meaningfully.

Notes

2.6 Check Yourself

In this chapter you have explored the ways we express numbers. Historically, many numeration systems were used to express numbers in different ways. A place value numeration system such as the modern world now uses provides a far more efficient way to express numbers than ancient systems, such as the Roman numeral system. Our use of base ten is probably due to the fact that we have ten fingers. Other bases could be used. Because we are so familiar with base ten, however, working with other bases is useful in appreciating the difficulties children have in learning to use base ten, particularly when learning to operate with numbers in base ten.

Understanding place value and its role in the elementary school mathematics curriculum is crucial. Too many teachers think that teaching place value is simply a matter of noting which digit is in the ones place, which is in the tens place, etc. But it is only when students have a deep understanding of place value that they can make sense of numbers larger than 10 and smaller than 1, and understand how to operate on these numbers. Most arithmetic errors (beyond careless errors) are due to a lack of understanding of place value. Unfortunately, the algorithms we teach usually treat digits in columns without attending to their values, and students who learn these algorithms without understanding the place value of each digit are far more likely to make computational errors.

You should be able to work problems like those assigned and to meet the following objectives.

1. Discuss the advantages of a place value system over other ancient numeration systems.
2. Explain how the placement of digits determines the value of a number in base ten, on both sides of the decimal point.
3. Explain how the placement of digits determines the value of a number in any base, such as base five or base twelve and answer questions such as: What does 346.3 mean in base twelve? Convert that number to base ten.
4. Given a particular base, write numbers in that system beginning with one.
5. Make a drawing with base materials that demonstrates a particular addition or subtraction problem, e.g., 35.7 + 24.7 or 35.7 – 24.7 in base ten.
6. Write base ten numbers in another base, such as 9 in base nine, or 33 in base two.
7. Add and subtract in different bases.
8. Understand the role of the unit, one, in reading and understanding decimal numbers.
9. Discuss problems that children who do not have a good understanding of place value might have when they do computation problems.

Notes

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