Chapter 20

Size Changes and Similarity

One important feature of congruent figures is that corresponding parts of the figures have exactly the same geometric measurements. In many applications of mathematics, however, two figures may have the exact same shape but not the same size. For example, photographic enlargements or reductions should look the same, even though corresponding lengths are different. This chapter gives a precise meaning to the idea of same shape, first with two-dimensional shapes and then with three-dimensional shapes.

20.1 Size Changes in Planar Figures

A moment’s thought about, say, a photograph of a building and an enlargement of that photograph, makes clear that for the buildings to look alike, corresponding angles have to be the same size. Activity 1 below will help us confront the less visible aspect of exact same shape. That is, how are corresponding lengths related?

Activity 1 A Puzzle About a Puzzle

Special tools needed: Ruler with metric (protractor optional)

In the diagram on the following page, the pieces may be cut out and then, as a puzzle, reassembled to make the square. You are to make a puzzle shaped just like the one given, but larger, using the following rule: The segment that measures 4 cm in the original diagram should measure 7 cm in your new version.

If you work as a group, each person should make at least one piece. When your group finishes, you should be able to put the new pieces together to make a square.

Continue on the next page.
Discussion 1  Students Discuss Their Methods

Here are some students’ descriptions of their thinking for Activity 1. They have agreed that the angles have to be the same, and they plan to use the many right angles to make the larger puzzle. Discuss each student’s thinking.

Lee: “7 is 3 more than 4. So you just add 3 to each length. 3 centimeters. Add 3 centimeters to 5 centimeters, and the new side should be 8 centimeters.”

Maria: “7 is $1 \frac{3}{4}$ times as much as 4, so a new length should be $1 \frac{3}{4}$ times the old length.”

Nerida: “To be the same-shaped puzzle, it’s got to be proportional to look the same. But I’m not sure how to make it proportional. Do you use ratios?”

Olivia: “From 4 centimeters to 7 centimeters is 75% more, so I would add 75% to each length. For example, take a 5-centimeter length; 75% of 5 centimeters is 3.75 centimeters, so the new length would be 5 + 3.75, or 8.75, centimeters.”

Pat: “If 4 centimeters grow to 7 centimeters, each centimeter must grow to $1 \frac{3}{4}$ centimeters. So 5 centimeters should grow to 5 times as much, that is, 5 times $1 \frac{3}{4}$ centimeters, and 6 cm should grow to 6 times $1 \frac{3}{4}$ centimeters.”
Activity 2  Super-Sizing It More
Make a sketch, share the work, and indicate all the measurements needed to get a bigger puzzle, where a 5 cm segment in the original square in Activity 1 measures 8 cm in the enlarged version. Then discuss your thinking with others.

Activity 3  Reducing
1. Make a sketch and indicate the measurements needed to get a smaller puzzle, where a 6 cm segment in the original square in Activity 1 should measure 4 cm on the smaller version.
2. In making an enlargement or reduction of a shape, as was done with the puzzle, how do angle sizes in the new shape compare with the corresponding ones in the original shape? How do lengths in the new shape compare with the corresponding ones in the original shape?
3. Write a set of instructions for enlarging/reducing such puzzles. Give a warning about any method that does not work, and explain why it does not work.

THINK ABOUT…
Which of the images on the next page would be acceptable as a reduced size of the given original drawing?

Original

Continue on the next page.
Similarity

Enlargements or miniatures must have the exact same shape as the original. Two shapes related this way are called similar in mathematics, in the technical sense of the word similar and not just because they are the same general shape.

Two shapes are similar if the points in the two shapes can be matched so that (1) every pair of corresponding angles have the same size, and (2) the ratios from every pair of corresponding lengths all equal the same value, called the scale factor.

The second point makes clear that it is the multiplicative comparisons of corresponding lengths, not the additive comparisons, that are crucial. This point can also be expressed in different but equivalent ways:

- \( \frac{\text{new length}}{\text{corresponding original length}} = \text{scale factor} \)
- \( \frac{\text{new length}}{\text{corresponding original length}} = \text{scale factor} \)
- \( \text{(new length)} = (\text{scale factor}) \times (\text{corresponding original length}) \)

The last version makes explicit the multiplicative effect of the scale factor.

Do you see that the scale factor for the original puzzle enlargement in Activity 1 is \( 1\frac{1}{4} \)? In most situations, which of the shapes is the new and which is the original (or old, if you prefer) is arbitrary. So long as the scale factor and ratios are interpreted consistently, the choice of new and original can be made either way. For example, if the lengths of the sides of a figure \( X \) are 4 times as long as those of another figure \( Y \), then the lengths of the sides of figure \( Y \) are \( \frac{1}{4} \) as long as those of figure \( X \).
So, if you are careful about keeping corresponding angles the same size and corresponding lengths related by the same scale factor, you can make a polygon similar to a given one. Another method, called the **ruler method**, for obtaining a similar polygon is given below. Try using this method on separate paper. You will need to choose your own point for a center, your own scale factor, and your own original polygon (any triangle or quadrilateral will do). The importance of the scale factor is apparent in this method.

**Steps in the Ruler Method for Size Changes**

1. Pick a point (which becomes the **center** of the size change). Draw a ray from the center through a point on the original shape. Measure the segment from the center to the point on the original shape.

   ![Diagram of Step 1](image)

2. Multiply that measurement by your chosen scale factor (we’ll use 1.7 here, so $1.7 \times 4 = 6.8$ cm).

3. Measure that distance (6.8 cm) from the **center** (that is important), along the ray starting at the center and going through the selected point. This distance gives what is called the **image** of the point.

   ![Diagram of Step 2](image)

4. Repeat Steps 1–3 with the other vertices of the original polygon. Connect all the images with the ruler. The resulting polygon is the **image** of the original polygon.

   ![Diagram of Step 4](image)
For a curved figure, the ruler method is not efficient because you have to go through the steps for too many points. But the ruler method works well for a figure made up of line segments. Starting with a more elaborate figure and then coloring the figure and its enlargement or reduction, can make an attractive display. The completed drawing often carries a three-dimensional effect.

**Activity 4 Oh, I See!**

a. Measure the lengths of the pairs of corresponding sides of the two quadrilaterals \( PQRS \) and \( P'Q'R'S' \) shown on the previous page to see how they are related.

b. How else do the corresponding sides appear to be related?

c. How are the pairs of corresponding angles of the polygons related in size?

**Size changes**, or **size transformations**, like the one shown in the ruler-method steps, are a basic way of getting similar polygons. After the size has changed, the image can be moved around by rotating or reflecting it, for example. Figure 1 gives an example. Triangle \( A'B'C' \) is similar to triangle \( ABC \) because it is the image of triangle \( ABC \) from the size change. Triangle \( A''B''C'' \) is a reflection of triangle \( A'B'C' \) about the line of reflection shown, and it is still similar to the original triangle. (Note the use of the \( A, A' \) and \( A'' \) to make corresponding points clear.) If a rotation or a reflection is involved, finding corresponding vertices and sides may require some attention.

![Figure 1](image)

Let us consider some examples that illustrate how all these general results are useful in dealing with similar triangles.
EXAMPLE 1

Make a rough sketch of triangles similar to the original triangle below, (a) using a scale factor of 2.5 and (b) using a scale factor of $\frac{4}{5}$. Also, find the sizes of the angles and sides of the triangles of each new triangle.

![Original Triangle](image)

SOLUTION

The size of the angle at J can be calculated by finding $180 - (40 + 86)$, which gives $54^\circ$. For either part (a) or (b), the angles in the similar triangles will be the same sizes as those in the original triangle: $40^\circ$, $86^\circ$, and $54^\circ$.

(a) Your sketch should show a larger triangle, with sides about 2.5 times as long as those in the original. The lengths of the sides of the similar triangle will be $2.5 \times 12.3$ for the new $\overline{JL}$, $2.5 \times 8$ for the new $\overline{JK}$, and $2.5 \times 10$ for the new $\overline{LK}$, all in meters. That is, the lengths will be (about) 30.8 m, 20 m, and 25 m.

(b) In the same way, the lengths will be 9.8 m, 6.4 m, and 8 m. Your sketch of the image triangle should be smaller, because the scale factor is less than 1.

![Image Triangle](image)

EXAMPLE 2

Suppose you are told that the following two quadrilaterals on the next page are similar. Find the missing angle sizes and lengths of sides.

Continue on the next page.
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SOLUTION

We are told that the two shapes are similar, so the first thing to do is to determine the correspondence. How the shapes are drawn helps, although the given angle sizes allow just one possibility. Because corresponding angles are the same size, the angle at W has size 85˚, and the angle at C has size 95˚.

To determine the missing lengths, we need the scale factor. The only pair of corresponding sides for which we know measurements are \( \overline{WZ} \) and \( \overline{AD} \). Thinking of ABCD as the original (it is usually easier to deal with scale factors greater than 1), the scale factor is 9:6, or \( \frac{9}{6} = 1.5 \). So then the length of \( \overline{WX} \) will be 1.5 \( \times \) 11.8, or 17.7 cm. To find the missing lengths in ABCD, we can solve 8 = 1.5 \( \times \) BC, and 12 = 1.5 \( \times \) DC, or we can reverse the viewpoint to make WXYZ the original, and work with a revised scale factor of 6:9, or \( \frac{2}{3} \). In either case, we find that \( \overline{BC} \) has length about 5.3 cm, and \( \overline{DC} \) is 8 cm long.

EXAMPLE 3

You are told that the two triangles below are similar, but they are deliberately not drawn to scale. Using the given measurements, find the missing lengths and angle sizes.

SOLUTION

The missing angle sizes present no problem because angles \( a \) and \( b \) are corresponding and the angle sum for a triangle is 180˚, so \( a \) and \( b \) are both 180 – (90 + 37), or 53˚.
The complication here for finding the scale factor is that the correspondence is not obvious: Does the 6 km correspond to the 20 km or to the 15 km side? Because the 6 km side is opposite the 37° angle, its correspondent in the other triangle should be opposite the 37° angle there. That would make 6 km and 15 km corresponding. (Alternatively, the 6 km side is common to the right angle and the 53° angle, so find those angles in the other triangle and use their common side.)

Similarly, the 10 km and y km sides correspond, as do the x km and 20 km sides. Using the scale factor \( \frac{15}{6} = 2.5 \), we find that

\[
y = 2.5 \times 10 = 25 \text{ cm}
\]

\[20 = 2.5 \times x, \text{ or } x = \frac{20}{2.5} = 8 \text{ cm}.
\]

It is slightly digressive, but let us review some language for which everyday usage is often incorrect when describing similarity and other situations. One of the segments in the illustration of the ruler method is about 2 cm, with its image about 3.4 cm. The comparison of the 2 cm and the 3.4 cm values can be correctly stated in several ways:

- “The ratio, 3.4 : 2, is 1.7.” (a multiplicative comparison)
- “3.4 cm is 1.7 times as long as 2 cm.” (a multiplicative comparison)
- “3.4 cm is 170% as long as 2 cm.” (a multiplicative comparison)
- “3.4 cm is 1.4 cm longer than 2 cm.” (an additive comparison and a true statement, but not the important one for similarity; note the -er ending on “longer”)

**Discussion 2**  
**Saying It Correctly**

Who is correct, Arnie or Bea? Explain. (A sketch might help, identifying the longer than part.)

Arnie stated, “3.4 cm is 1.7 times longer than 2 cm.”

Bea argued, “3.4 cm is 1.7 times as long as 2 cm, but 3.4 cm is only 0.7 times longer than 2 cm, or 70% longer than 2 cm.”

Incorrect language can be heard especially when both additive and multiplicative languages are used in the same sentence. Most people, however, do correctly fill in the blanks in statements such as “__ is 50% as big as 10” and “__ is 50% bigger than 10,” so these examples might be helpful as checks in other sentences.

**TAKE-AWAY MESSAGE** . . . When you want to show that two shapes are indeed similar, you need to confirm these two conditions: (1) Corresponding angles must have the same size, and (2) the lengths of every pair of corresponding segments must have the same ratio, that is, the scale factor. Vice versa, knowing that two figures are similar tells you that both these conditions have been met, which allow you to determine many missing measurements in similar figures. Using the correct language in comparing lengths in similar shapes requires some care.
Learning Exercises for Section 20.1

Have your ruler and protractor handy for some of these exercises.

1. a. Summarize how the following are related, for a polygon and its image under a size change: corresponding lengths, corresponding angle sizes.
   b. How do the perimeters of similar polygons compare? Explain your thinking.

2. Tell whether the two shapes given in each part are similar. How do you know?
   a. A 6 cm by 7 cm rectangle, and a 12 cm by 13 cm rectangle

   b. 

      \[
      \begin{array}{c}
      \text{2 cm} \\
      \text{2 cm} \\
      \text{2 cm} \\
      \text{2 cm} \\
      \text{3 cm} \\
      \text{3 cm}
      \end{array}
      \quad \text{and} \quad 
      \begin{array}{c}
      \text{3 cm} \\
      \text{3 cm} \\
      \text{3 cm}
      \end{array}
      \]

3. Copy and finish the incomplete second triangle to give a similar triangle, so that the 2 cm and 5 cm sides correspond. Does your triangle look similar to the original one?

4. With a ruler, draw a triangle and find its image for a size transformation with the scale factor 4 and with center at a point of your choice. Plan ahead so that the image will fit on the page.

5. With a ruler, draw a trapezoid and find its image for a size transformation with the scale factor 2.4 and with center at a point of your choice.

6. With a ruler, draw a triangle and find its image for a size transformation with the scale factor \( \frac{3}{4} \) and with center at a point of your choice.

7. a. Measure the angles and sides of your polygons in Learning Exercises 4, 5, and 6. Verify the key relationships about lengths and angles in size changes that you summarized in Learning Exercise 1(a).
   b. Besides the ratio of lengths, how do a side of a polygon and its image appear to be related in the ruler method for size changes?
   c. Check the key relationships and your ideas about angles and sides from part (b) on the two similar triangles given on the next page.
8. Find the sizes of all the angles and sides of shapes that are similar to the original parallelogram below with the scale factors:
   a. 6.1  
   b. $\frac{2}{3}$

![Parallelogram with measurements](image)

   c. What shape are the images in parts (a) and (b)?

9. Find the scale factor and the missing measurements in the similar triangles in each part. (The sketches are not drawn to scale.)

   a. 
   
   ![Similar Triangle A](image)

   ![Similar Triangle B](image)

   ![Similar Triangle C](image)

   b. 

   ![Similar Triangle A'](image)

   ![Similar Triangle B'](image)

   ![Similar Triangle C'](image)

   c. 

   ![Similar Triangle A"](image)

   ![Similar Triangle B"](image)

   ![Similar Triangle C"](image)

*Continue on the next page.*
10. Can the center of a size transformation be
   a. inside a figure?  
   b. on the figure?

11. For a given scale factor and a given figure, what changes if you use a different point for the center of a size transformation?

12. For a given center and a given figure, what changes if you use a different scale factor for a size transformation?

13. Suppose the scale factor is 1 for a size transformation. What do you notice about the image?

14. Scale factors often are restricted to positive numbers.
   a. What would the image of a figure be if a scale factor were allowed to be 0?
   b. How could one make sense of a negative scale factor, say, -2?

15. Is each of the following sentences phrased correctly? Correct any that is not by changing a number.
   a. 60 is 200% more than 30.
   b. 12 cm is 150% longer than 8 cm.
   c. $75 is 50% more than $50.
   d. The 10K run is 100% longer than the 5K.

16. a. Janeetha said, “I increased all the lengths by 60%.” If Janeetha is talking about a size change, what scale factor did she use?
   b. Juan used a scale factor of 225% on a 6 cm by 18 cm rectangle. How many centimeters longer than the original dimensions are the dimensions of the image? How many percent longer are they than in the original?

17. Consider this original segment: __________
   Draw another segment that fits each description.
   a. 3 times as long as the original segment
   b. 1.5 times longer than the original segment
   c. 300% longer than the original segment

   Consider this original region: __________
   Draw another region that fits each description.
   d. 3 times the area of the original region
   e. 1.5 times the area of the original region
   f. 300% more than the area of the original region
Section 20.1  Size Changes in Planar Figures

18. In each part, which statements express the same relationship? Support your decisions with numerical examples or sketches.
   a. “This edge is 50% as long as that one,” vs. “This edge is 50% longer than that one,” vs. “This edge is half that one.”
   b. “This quantity is twice as much as that one,” vs. “This quantity is 200% more than that one,” vs. “This quantity is 100% more than that one.”
   c. “This value is 75% more than that one,” vs. “This value is \( \frac{3}{4} \) as big as that one,” vs. “This value is \( 1 \frac{1}{4} \) times as big as that one,” vs. “This value is 75% as much as that one.”

19. In each part, give a value that fits each description.
   a. \( 2 \frac{1}{3} \) times as long as 24 cm
   b. \( 2 \frac{1}{3} \) times longer than 24 cm
   c. 75% as long as 24 cm
   d. 75% longer than 24 cm
   e. 125% more than 24 cm
   f. 125% as much as 24 cm
   g. 250% as large as 60 cm
   h. 250% larger than 60 cm

20. How would you make a shape similar to the following parallelogram with the scale factor 4? Give two ways.

![Parallelogram](image)

21. Explain how size transformations are involved in each of the following situations.
   a. photographs
   b. different maps of the same location
   c. model cars or architectural plans
   d. banking interest (This won’t involve shapes!)

22. What is the scale of a map if two locations 3 inches apart on the map are actually 84 miles apart in reality?
23. The following diagram shows two maps with the same two cities, River City and San Carlos. Even though the second map does not have a scale, determine the straight-line distance from San Carlos to Beantown.

Map 1
River City

San Carlos

Map 2
River City

San Carlos

1 cm = 30 km

24. Timelines are representations that also use scales. Make a time line 20 cm long, starting at year 0, and mark the following dates:

- Magna Carta 1215
- Columbus 1492
- Declaration of Independence 1776
- French Revolution 1789
- Civil War 1861–1865
- Wright brothers’ flight 1903
- World War II 1941–1945
- First atomic bomb 1945
- Commercial television 1950s
- Personal computers late 1970s
- Your birth

Add any other dates you wish.

25. a. Make a timeline 20 cm long to represent the following geologic times.
   - Cambrian, 600 million years ago (first fossils of animals with skeletons)
   - Carboniferous, 280 million years ago (insects appear)
   - Triassic, 200 million years ago (first dinosaurs)
   - Cretaceous, 65 million years ago (dinosaurs gone)
   - Oligocene, 30 million years ago (modern horses, pigs, elephants, and so on, appear)
   - Pleistocene (first humans, about 100,000 years ago)

b. If you were to add the Precambrian, 2 billion years ago (first recognizable fossils), and use the same scale as in part (a), how long would your time line have to be?

26. Suppose a rectangle undergoes a size change with scale factor 3, and then that image undergoes a second size change, with scale factor 4. Are the final image and the original rectangle similar? If so, what is the scale factor?
27. a. Some teachers like to use two sizes of grids and have students make a larger or smaller version of a drawing in one of the grids. Try this method, as shown.

b. What scale factor is involved in part (a)? (Hint: Measure.)
c. Can this method be used to make a smaller image? Explain how or why not.

28. a. Equilateral triangle X has sides 7 cm long, and equilateral triangle Y has sides 12 cm long. Are X and Y similar? Explain.
b. Are two arbitrarily chosen equilateral triangles similar? Explain.
c. Are every two right triangles similar? Explain.
d. Are every two squares similar? Explain.
e. Are every two rectangles similar? Explain.
f. Are every two hexagons similar? Explain.
g. Are every two regular \(n\)-gons (with the same \(n\)) similar? Explain.

29. Some reference books show pictures of creatures and give the scale involved. Find the actual sizes of these creatures. (Suggestion: Use metric units.)
a. Scale factor is 1:170.
b. Scale factor is 7.3:1.
30. In the drawing below, \( x' \) and \( x'' \) are the images of \( x \) for size transformations with center \( C \) and the respective scale factors \( r \) and \( s \). (These relationships are fundamental in trigonometry.)

![Diagram of size transformations](image)

a. Find these ratios: \( \frac{x'}{y'}, \frac{x''}{y''}, \frac{y'}{z'}, \text{ and } \frac{y''}{z''} \).

b. How do these ratios compare: \( \frac{x}{y}, \frac{x'}{y'}, \text{ and } \frac{x''}{y''} \)?

c. How do these ratios compare: \( \frac{y}{z}, \frac{y'}{z'}, \text{ and } \frac{y''}{z''} \)?

### 20.2 More About Similar Figures

You know two methods for creating similar figures: (1) Apply the two criteria (make corresponding angles the same size, and use the same scale factor in changing the lengths), and (2) use the ruler method for performing a size change. This section discusses other interesting results that arise once you have similar figures and a very easy way of knowing that two triangles are similar.

**Activity 5 Finding Missing Measurements**

1. Suppose that original triangle PQR below is similar to triangle P'Q'R' (not shown) with scale factor 5. What are the sizes of the angles and sides of triangle P'Q'R'? The units for the lengths are kilometers (km).

![Diagram of triangles PQR and P'Q'R']

2. Find the perimeters (distance around) of triangles PQR and P'Q'R' and compare them.
**THINK ABOUT…**

Why are the perimeters of similar polygons also related by the scale factor?

Although the reason may be difficult to put in words, a form of the distributive property—for example, $5p + 5q + 5r = 5(p + q + r)$—gives a mathematically pleasing justification. Do you see the two perimeters in the equation?

With lengths and perimeters of similar shapes related by the scale factor, a natural question is: How is the area of a polygon related to the area of its image, for a size change? With a size change drawing, you can conjecture the answer *without having to figure out the values of the two areas*! Examine the following figure, which uses 2 as the scale factor; you can see the relationship without finding the area of either triangle.

**THINK ABOUT…**

For a size change with scale factor 3, the area of the image of a shape is ____ times as large as the area of the original shape. (Make a rough sketch, using a triangle.) Make a conjecture for size changes with other scale factors.

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**Determining Similarity of Triangles**

So far we have examined creating similar shapes (Section 20.1) and the relationships that exist when two shapes are known to be similar (Section 20.1 and the previous
Think Abouts). But how would you know whether two triangles are similar, especially if both could not be in your field of vision at the same time? After thinking about it, your first response would likely be, “Measure all the angles and sides in both triangles. See whether they can be matched so that corresponding angles are the same size and the ratios of corresponding lengths are all the same (which would give the scale factor).” And you would be correct. Indeed, you would have described a method that could be applied to polygons of any number of sides, not just triangles. But for triangles we are especially lucky; we need to find only two pairs of angles that are the same size.

Two triangles are similar if their vertices can be matched so that two pairs of corresponding angles have the same size.

The assertion is that the third pair of angles and the ratios of corresponding lengths take care of themselves. For example, if one triangle has angles of 65° and 38° and the other triangle does too, then even without any knowledge about the other angle and the sides, the two triangles must be similar. One triangle is the image of the other by a size change, along with possibly some sort of movement like a reflection or a rotation. But will you also know the scale factor? The answer is “No.” Finding the scale factor involves knowing the lengths of at least one pair of corresponding sides.

EXAMPLE 4

Using the information given in the following triangles, (a) tell how you know that they are similar, (b) find all of the missing measurements, and (c) give the ratio of their areas. The triangles are not drawn to scale.

SOLUTION

(a) The 45° pair and the 17° pair assure that the triangles are similar.

(b) It is easy to find the sizes of the third angles from the angle sum in a triangle: 118°. To find the missing lengths, we need the scale factor. The drawing here makes it easy to find corresponding sides. The known lengths 15 cm and 24 cm are corresponding lengths. Using the left-hand triangle as the original, we get the scale factor $\frac{24}{15} = 1.6$. Then $y$ is 19.2 cm, and $x$ is 5 cm (from $1.6x = 8$).

(c) The ratio of the areas is the square of the scale factor: $(1.6)^2 = 2.56$. The larger triangle has an area that is 2.56 times as large as that of the smaller triangle.
EXAMPLE 5

Given the information in the following drawings, find the missing measurements and give the ratio of the areas of the triangles. The triangles are not drawn to scale.

\[ \text{SOLUTION} \]

Angles \( a \) and \( b \) must have 53° because the right angles have 90° and the other given angles have 37°. The right angles and either the 53° angles or the 37° angles tell us that the two triangles are similar. Finding corresponding parts takes some care, and we need to know two corresponding lengths. Perhaps after other trials, we notice that the 6 mi and 9 mi segments both are opposite the 37° angles. So, the scale factor is 1.5. Using the scale factor, we find that \( y \) is 15, \( x \) is 8, and the ratio of the areas is \((1.5)^2\), or 2.25.

TAKE-AWAY MESSAGE . . . The ratio of the perimeters of similar figures is the same as the scale factor, but the ratio of their areas is the square of the scale factor. Justifying that two triangles are similar is easy because you need to find only two pairs of angles that are the same size. Finding the correspondence in two similar figures can require some care.

Learning Exercises for Section 20.2

1. Summarize how the following items are related for a polygon and its image under a size change: corresponding lengths, corresponding angle sizes, perimeters, and areas.

2. How would you convince someone that the ratio of the areas of two similar triangles is the square of the scale factor?

3. In each part, are the triangles similar? Explain how you know.

   a. 

   b. Triangle 1: angles 50°, 25° and triangle 2: angles 25°, 105°
4. In parts (a)–(d), find the missing lengths. Explain how you know the figures are similar and how you know which segments correspond to each other. (The sketches are not to scale.)

c. Triangle 1: angles 70°, 42° and triangle 2: angles 48°, 70°

d. Right triangle 1: angle 37° and right triangle 2: angle 53°

e. Give the ratio of the areas of the triangles in parts (a) and (c).

f. Devise a method for determining the width of a pond.

(Suggestion: See part (d).)
5. A rectangle 9 cm wide and 15 cm long is the image for some size change of an original shape having width 4 cm.
   a. What is the scale factor of the size transformation?
   b. What type of figure is the original shape?
   c. What are the dimensions of the original shape?
   d. What are the areas of the original shape and the image? How are they related?
   e. If the description, width 4 cm, in the original description was replaced by one dimension 4 cm, which of parts (a)–(d) could be answered differently? (Hint: What are the names for the dimensions of a rectangle?)

6. If the two triangles in the diagram below are related by a size transformation, find the scale factor. How many centimeters is \( x \)? (This sort of diagram is common in the study of light and lenses.)

![Diagram of two triangles with dimensions 60 cm, 40 cm, and 30 cm]

7. a. Your archaeological exploration has found a huge stone monument with the largest face being a triangular region. It appears to have the same proportions as a monument at another location. You telephone someone at the other location. What questions would you ask, at minimum, to determine whether the two triangles are similar?
   b. Suppose the situation in part (a) involves quadrilaterals. What questions would you ask, at minimum, to determine whether the two quadrilaterals are similar?

20.3 Size Changes in Space Figures

A size change for a three-dimensional shape figure is much like a size change for a two-dimensional shape. This section deals with the ideas and terminology as though they were new topics and should strengthen your understandings from earlier work.
Discussion 3  Related Shapes

Which of the shapes in the following figure would you say are related in some way? Is there another collection of the shapes that are related in some way?

Activity 6  Here's Mine

Make or sketch other shapes that you think would be related to shape B and to shape F above. Explain how they are related.

The rest of this section focuses on one of these relationships, similarity.

Discussion 4  Larger and Smaller

An eccentric $10^9$-aire owns an L-shaped building like the one below.
Section 20.3  Size Changes in Space Figures

a. She wants another building designed, “shaped exactly like the old one, but twice as large in all dimensions.” Make or draw a model shaped like this second building that she wants.

b. Below are some diagrams. Which, if any, of the following drawings will meet the criterion? Explain.

c. On isometric dot paper, show your version of a building (call it Building 2) that will meet the 10⁹-aire’s criterion, and compare your drawing with those of others. Discuss any differences you notice.

d. Now the 10⁹-aire wants a third building (call it Building 3) designed to be shaped like the original, but “three-fourths as large in all dimensions.” Make a drawing or describe this Building 3.

Buildings acceptable to the billionaire in Discussion 4 are examples of similar polyhedra. The word similar has a technical meaning and is used when one shape is an exact enlargement (or reduction) of another. What exact means will come out of the next Think About.

**THINK ABOUT…**

Identify some quantities in the billionaire’s original building. How are the values of these quantities related to the values of the corresponding quantities in Building 2?

a. In particular, how are new and original lengths related?

b. How are new and original angles related?

c. How are new and original surface areas (the number of square regions required to cover the building, including the bottom) related?

d. How are new and original volumes (the number of cubical regions required to fill the building) related?

For a given pair of buildings, you may have noticed that the angles at the faces are all the same size. You also may have noticed that the ratio of a new length to the corresponding original length was the same, for every choice of lengths. That is, length in one shape

______________________

length in other shape

is the same ratio for all corresponding segments (assuming the ratios are formed in a consistent fashion, each ratio starting with the same building). This ratio is called the **scale factor** for the enlargement (or reduction). In a short form, we can write

\[
\frac{\text{new length}}{\text{original length}} = \text{scale factor}.
\]
An algebraically equivalent and useful form for similar figures is

\[(\text{new length}) = (\text{scale factor}) \times (\text{original length}).\]

With either form, if you know two of the values, you can find the third one. Notice that if the scale factor is \(k\), the last equation says that a new length is \(k\) times as long as the corresponding original length. The wealthy woman could have used the term \textit{scale factor} in her requests: “Building 2 should be built with scale factor 2, and Building 3 should be built with scale factor \(\frac{3}{4}\).” When two polyhedra are similar, every ratio of corresponding lengths must have the same value, and every pair of corresponding angles must be the same size. Because lengths are affected by the scale factor, it is perhaps surprising that angle sizes do \textit{not} change; that is, corresponding angles in similar shapes will have the same sizes.

**Two 3D shapes are similar if the points in the two shapes can be matched so that (1) every pair of corresponding angles have the same size, and (2) the ratios from every pair of corresponding lengths all equal the same value, called the scale factor.**

So, corresponding lengths and angle sizes in similar figures are related. The relationships between surface areas and between volumes for similar 3D shapes are important as well. From your results in the Think About, what conjectures are reasonable?

\[
\text{(new surface area)} = \text{________} \times (\text{original surface area})
\]

\[
\text{(new volume)} = \text{________} \times (\text{original volume})
\]

We end this section with this important point: For any scale factor, say, \(\frac{\text{new}}{\text{original}} = \frac{6}{7}\), the ratio \(\frac{6}{7}\) does \textit{not} necessarily mean that new = 6 and original = 7. For example, new could be 60 and original could be 70, but the ratio \(\frac{\text{new}}{\text{original}}\) would still = \(\frac{6}{7}\).

**TAKE-AWAY MESSAGE** . . . The following descriptions are all important relationships to know.

<table>
<thead>
<tr>
<th>Quantities</th>
<th>How 2D similar shapes are related.</th>
<th>How 3D similar shapes are related.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sizes of corresponding angles</td>
<td>equal</td>
<td>equal</td>
</tr>
<tr>
<td>Lengths of corresponding segments</td>
<td>ratios = scale factor</td>
<td>ratios = scale factor</td>
</tr>
<tr>
<td>Areas/surface areas</td>
<td>ratio = (scale factor)²</td>
<td>ratio = (scale factor)²</td>
</tr>
<tr>
<td>Volumes</td>
<td>(not applicable)</td>
<td>ratio = (scale factor)³</td>
</tr>
</tbody>
</table>
Learning Exercises for Section 20.3

1. Summarize the relationships among length and angle measurements in similar polyhedra. In particular, how is the scale factor involved? How are the areas of similar polyhedra related? The volumes?

2. **a.** Now the billionaire from Discussion 4 wants two more buildings sketched, each similar to the original one. Building 4 should have scale factor $\frac{1}{2}$, and Building 5 should have scale factor 2.5. Make sketches to show the dimensions of Buildings 4 and 5.
   
   **b.** What is the scale factor between Building 4 and Building 5?

3. Are any of the following shapes similar? Explain your decisions, and if two shapes are similar, give the scale factor. (Make sure that it checks for every dimension.)

4. Which, if any, of the following shapes are related? Explain.

   **h.** How many shapes like shape (b) would it take to make shape (a)?
5. Are any of the following right rectangular prisms similar to a right rectangular prism with dimensions 3 cm, 7 cm, and 8 cm (in other words, a 3 cm by 7 cm by 8 cm or a 3 cm × 7 cm × 8 cm one)? If they are similar, what scale factor is involved (that is, is every ratio of corresponding lengths the same)? Explain your decisions, including a reference to corresponding angles of the two prisms.
   a. 5 cm × 9 cm × 10 cm
   b. 96 cm by 36 cm by 84 cm
   c. 8.7 cm × 20.3 cm × 23.2 cm
   d. 6 cm by 14 cm by 14 cm
   e. 3 inch × 7 inch × 8 inch
   f. 5 cm by 11.67 cm by 13.33 cm
   g. 9 cm × 49 cm × 64 cm
   h. 7 cm × 14 cm × 15 cm
   i. 15 mm × 35 mm × 4 cm
   j. Are the prisms in parts (b) and (c) similar?

6. a. A detailed model of a car is 8 inches long. The car is actually 12 feet long. If the model and the car are similar, what is the scale factor?
   b. A natural history museum has prepared a 12-foot long model of one kind of locust. They say the model is 70 times life size. What is the life size of this locust?

7. a. What other measurements of lengths and angles do you know about the following right rectangular prisms P and Q, if they are similar?
   b. What scale factor is involved if P is the original shape and Q is the new one?
   c. What scale factor is involved if shape Q is the original and shape P the new one?

8. Olaf lives in a dorm in a tiny room that he shares with three others. He wants to live off campus next year with his friends, but he needs more money from his parents to finance the move. He decides to build a scale model of his dorm room so that when he goes home for break, he can show his parents the cramped conditions he lives in. He decides to let one inch represent 30 inches of the actual
Section 20.3  Size Changes in Space Figures

lengths in the room. His desk is a right rectangular prism 40 inches high, 36 inches long, and 20 inches wide. He decides that his scaled desk should be \(1 \frac{1}{3}\) inches high, but a roommate says it should be 10 inches high. Who is right and why? What are the other dimensions of the scaled desk?

9. Make sketches to guide your thinking in answering the following questions.
   a. How many centimeter cubes does it take to make a 2 cm by 2 cm by 2 cm cube?
   b. How many centimeter cubes does it take to make a 3 cm by 3 cm by 3 cm cube?
   c. Are the two large cubes in parts (a) and (b) similar? Explain.
   d. How do the surface areas of the two cubes compare? There are two ways to answer: (1) Direct counting of squares to cover, and (2) the theory of how the scale factor is involved.
   e. How do the volumes of the two cubes compare?

10. a. Is a polyhedron similar to itself (or to a copy of itself)? If not, explain why. If so, give the scale factor.
   b. If polyhedron X is similar to polyhedron Y, is Y similar to X? If not, explain why. If so, how are the scale factors related?
   c. Suppose polyhedron X is similar to polyhedron Y and polyhedron Y is similar to polyhedron Z. Are polyhedra X and Z similar? If not, explain why. If so, how are the scale factors related?

11. a. Give the dimensions of a right square prism that would be similar to one with dimensions 20 cm, 20 cm, and 25 cm.
   b. What scale factor did you use?
   c. How does the total area of all the faces of the original prism compare with that of your prism? A rough sketch may be useful.
   d. How do the volumes of the two prisms compare?

12. Repeat Learning Exercise 11, letting the scale factor be \(r\).

13. Why is it ambiguous to say, “This polyhedron is twice as large as that one”?

14. Give the dimensions of a right rectangular prism that would be similar to one that is 4 cm by 6 cm by 10 cm with
   a. scale factor 2.2.
   b. scale factor 75%.
   c. scale factor \(\frac{4}{7}\).
   d. scale factor \(\frac{3}{7}\).
   e. scale factor 100%.
   f. scale factor 220%.

15. Suppose a 2 cm by 3 cm by 5 cm right rectangular prism undergoes a size transformation with scale factor 360%. What are the surface area and volume of the image of the prism? What are its dimensions?

16. You have made an unusual three-dimensional shape from 8 cubic centimeters and want to make another one five times as big for a classroom demonstration. If five times as big refers to lengths, how many cubic centimeters will you need for the bigger shape? Explain your reasoning.
17. Can a scale factor be 0? Explain.

18. Are the following shapes similar? If not, explain why. If so, tell how you would find the scale factor.
   a. a cube with 5 cm edges and a cube with 8 cm edges
   b. every two cubes, with their respective edges $m$ cm and $n$ cm long
   c. every two triangular pyramids
   d. every two right rectangular prisms
   e. a rhomboidal prism with all edges $x$ cm long and a cube with edges $3x$ cm long

19. Why can’t a cube be similar to any pyramid?

20. Given a net for a polyhedron, how would you make a net that will give a larger (or smaller) version of that polyhedron?

21. One student explained the relationship between the volumes of two shapes this way: “Think of each cubic centimeter in the original shape. It grows to a $k$ by $k$ by $k$ cube in the enlargement. So, each original cubic centimeter is now $k^3$ cubic centimeters.” Retrace her thinking for two similar shapes related by the scale factor 4, using a drawing to verify that her thinking is correct.

22. Suppose the scale factor relating two similar polyhedra is 8.
   a. If the surface area of the smaller polyhedron is 400 cm$^2$, what is the surface area of the larger polyhedron?
   b. If the volume of the smaller polyhedron is 400 cm$^3$, what is the volume of the larger polyhedron?

23. Polyhedron 1, which is made up of 810 identical cubes, is similar to Polyhedron 2, which is made up of 30 cubes of that same size.
   a. What is the scale factor relating the two polyhedra?
   b. What is the ratio of the surface areas of the two polyhedra?

24. Legend: Once upon a time there was a powerful but crabby magician who was feared by her people. One year she demanded a cube of gold, 1 meter on an edge, and the people gave it to her. The next year she demanded, “Give me twice as much gold as last year.” When they gave her a cube of gold, 2 meters on an edge, she was furious—“You disobedient people!”—and she cast a spell over all of the people. Why? And why should she have been pleased?

25. Make two nets for a cube so that the nets are similar as 2D shapes but also so that one net will have an area four times as large as that of the first net. When the nets are folded to give cubes, how will the surface areas of the two cubes compare? (Hint: How will the lengths of the edges compare, in the two nets?)

26. The index finger of the Statue of Liberty is 8 ft long. Measure the length of your index finger, the length of your nose, and the width of your mouth. Use the information to predict the length of the Statue of Liberty’s nose and the width of her mouth. What are you assuming?
Section 20.4 Issues for Learning: Similarity

Similarity often comes up in the intermediate grades in the elementary school mathematics curriculum, but perhaps just as a visual exercise. Students are asked “Which have exactly the same shape?” for a collection of drawings. Occasionally there is a little numerical work, usually with scale drawings and maps (and the latter may not be associated with similarity in the children’s minds). Although the overall situation may be changing, there is much less research on children’s thinking in geometry than on number work, with only a scattering of studies dealing with similar figures. Here is a task that has been used in interviews of children of various ages.

(A drawing like the one to the right is given to the student, along with a chain of paper clips.) Mr. Short is 4 large buttons in height. Mr. Tall (deliberately not shown to the student) is similar to Mr. Short but is 6 large buttons in height. Measure Mr. Short’s height in paper clips (he is 6 paper clips tall in the drawing actually used in the interviews) and predict the height of Mr. Tall if you could measure him in paper clips. Explain your prediction.

Would you be surprised to learn that more than half the fourth graders (and nearly 30% of the eighth graders) would respond something like this? “Mr. Tall is 8 paper clips high. He is 2 buttons higher than Mr. Short, so I figured he is two paper clips higher.” Plainly, the students noticed the additive comparison of 6 buttons with 4 buttons, but they did not realize that it is the multiplicative comparison, the ratio, that is important for similar shapes. The younger children, of course, may not have dealt with similarity, and there may be developmental reasons why numerical work with similarity does not come up earlier in the curriculum. But the older students most likely had experienced instruction on proportions, yet they had not fully understood the idea of similar figures and/or the relevance of the ratio relationship in the Mr. Short-Mr. Tall task.

You, or someone else in your class, may have focused on the additive comparison in working with Activity 1 (A Puzzle About a Puzzle) so the lack of recognition of the importance of multiplicative comparisons for similar shapes clearly can continue beyond grade eight.
Chapter 20  Size Changes and Similarity

TAKE-AWAY MESSAGE . . . Additive comparisons seem to be natural, perhaps from many occurrences outside of school, but multiplicative relationships, even beyond those in similarity of shapes, may require schooling.

20.5 Check Yourself

You should be able to work problems like those assigned and to meet the following objectives.

1. Appreciate that creating an enlargement or reduction (a size change) of a shape involves a particular relationship among any pair of corresponding lengths.

2. More completely, know the criteria for two figures to be similar (the two angles in every pair of corresponding angles have the same size, the lengths of every pair of corresponding segments are related by the same scale factor).

3. Use the two criteria for similarity in determining missing angle sizes and lengths in similar shapes. Applications involving similarity (for example, photographs, maps, scale drawings, and time lines) might come up.

4. Create similar polygons with the ruler method (Section 20.1).

5. Distinguish between, and use correctly, such phrases as “times as long as” versus “times longer than,” or “85% as big as” versus “85% bigger than.”

6. State and use the relationships between the perimeters and the areas of two similar figures.

7. Use the work-saving way of telling whether two triangles are similar.

8. Extend the ideas of similarity with 2D figures to 3D shapes. That is, be able to tell whether two given 3D shapes are similar, and for 3D shapes that are known to be similar, find missing angle sizes, lengths, surface areas, and volumes. State the relationships between the surface areas and volumes of similar 3D shapes.

9. Illustrate a difficulty that children may have with multiplicative thinking (Section 20.4).

REFERENCE FOR CHAPTER 20