## Summary of Formulas

<table>
<thead>
<tr>
<th>Formula</th>
<th>Reasoning</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Sum of angle sizes for n-gon</strong> = ((n - 2)\times 180^\circ)</td>
<td>Split polygon with (n) sides into ((n - 2)) triangles.</td>
</tr>
<tr>
<td><strong>A(rectangular region)</strong> = (lw)</td>
<td>(w) rows</td>
</tr>
<tr>
<td><strong>Special case:</strong> (A)(square region) = (s^2)</td>
<td>(l) squares in each row</td>
</tr>
<tr>
<td><strong>A(parallelogram region)</strong> = (bh)</td>
<td></td>
</tr>
<tr>
<td><strong>A(triangular region)</strong> = (\frac{1}{2}bh)</td>
<td></td>
</tr>
<tr>
<td><strong>A(trapezoid region)</strong> = (\frac{1}{2}h(a + b))</td>
<td></td>
</tr>
<tr>
<td><strong>A(circular region)</strong> = (\pi r^2)</td>
<td>Area: Based on the total area of infinitely many triangles (Section 25.1)</td>
</tr>
<tr>
<td><strong>C(circle)</strong> = (2\pi r = \pi d)</td>
<td>Circumference: Similarity (Section 25.1)</td>
</tr>
<tr>
<td><strong>Surface area of a polyhedron</strong> = sum of the areas of its faces</td>
<td>Key idea: Measure an object by cutting into pieces, measuring each piece, and adding.</td>
</tr>
<tr>
<td><strong>(V = Bh)</strong> for any prism or cylinder with base (B) and height (h)</td>
<td>Base (B) numerically gives the number of cubes in one layer; (h) gives the number of layers.</td>
</tr>
<tr>
<td><strong>(V = \frac{1}{3}Bh)</strong> for any pyramid or cone with base (B) and height (h)</td>
<td>Suggested by experiment; no general justification given in this book</td>
</tr>
<tr>
<td><strong>A(surface of sphere)</strong> = (4\pi r^2)</td>
<td>Area formula: No justification given in this book</td>
</tr>
<tr>
<td><strong>(V = \frac{4}{3}\pi r^3)</strong> for any sphere</td>
<td>Volume formula: Based on the total volume of infinitely many pyramids (Section 25.2)</td>
</tr>
</tbody>
</table>
Glossary

Terms that are related to some noun or are adjectives may often be found with the associated noun. Numbers like [16.2] refer to sections. Some terms first appear in exercise lists, but are nonetheless important. The web site, http://nw.pima.edu/dmeeks/spandict/, currently gives English to Spanish translations of many mathematical terms.

**altitude.** . .of a triangle: a line segment from a vertex of the triangle that makes a right angle with the opposite side; also, the length of that segment. [x]

**of a trapezoid or parallelogram:** a line segment making right angles with each of two parallel sides; also, the length of that segment. [xx]

**of a pyramid or cone:** the line segment from the vertex of the pyramid or cone that makes a right angle with every line in the base that it meets; also, the length of that segment. [xx]

**of a prism or cylinder:** a line segment from the plane of one base, to the other base, so that it is perpendicular to every line in the other base that it meets; also, the length of the segment. [xx]

**angle:** two views—(1) two rays from the same point, or (2) a ray along with the result of its turning on its endpoint to a final position; the common point is its **vertex**, and the rays are its **sides**. [17.1, 23.2]

**acute angle:** an angle smaller than a right angle (has size less than 90˚). [17.1]

**adjacent angles:** angles with a common side between them. [17.1]

**alternate interior angles:** angles like those marked $x$ and $x'$ in the drawing below, also those marked $y$ and $y'$. [23.2 Learning Exercises]

**central angle:** an angle with its vertex at the center of a circle. [21.1]

**complementary angles:** two angles with sizes that sum to 90˚. [17.1]

**corresponding angles:** a pair of angles like those marked $a$ in the drawing below, also those marked $b$ or $c$ or $d$; in congruent or similar shapes, angles that match. [23.2 Learning Exercises]

**dihedral angle:** two half-planes from the same line. [23.2 Learning Exercises]

**exterior angle** of a polygon: an angle formed by one side of polygon with the extension of another side that passes through an endpoint. [17.1]

**inscribed angle:** an angle with its vertex on a circle, and its sides chords of the circle. [21.1]

**obtuse angle:** an angle with size between 90˚ and 180˚. [17.1]

**right angle:** an angle of 90˚, or representing a quarter of a full turn. [17.1]

**straight angle:** an angle with its sides along a straight line (has size 180˚). [17.1]

**supplementary angles:** two angles with sizes that sum to 180˚ (e.g., $x$ and $y$ below). [17.1]

**vertical angles:** the non-adjacent angles formed when two lines cross (for example, $a$ and $d$, or $b$ and $c$, below). [23.2 Learning Exercises]

![Diagram of angles and lines]
**arc** of a circle: see Circle. [21.1]

**area**: a metric system unit for area (100 square meters). [24.1]

**area** of a region: The number of square units that would be required to cover the region. The region could be a 2D region, or it could be the surface of a 3D figure, in which case it may be called the **surface area**. [24.1; formulas for determining areas, 25.1]

**axis** of a rotational symmetry of a 3D shape: the line about which the shape is rotated. [18.2]

**base** of a prism or pyramid: see **prism** or **pyramid**. [16.2]

**bisector** of an angle: a ray that cuts a given angle into two angles of the same size [construction of, 21.1]; of a line segment: a line that goes through the midpoint of the segment. [construction of, 21.1]

**capacity** of a 3D container: the **volume** that the container can hold. [24.2]

**center** of a rotational symmetry of a shape: the point about which the shape is rotated. [18.1]

**circle**: the 2D shape formed by all points at a given distance from a given point, called the **center** of the circle. Any segment from the center to the circle is a **radius** (plural: radii); **radius** may also refer to the length of such a segment. An **arc** of a circle is the piece of the circle between two points of the circle; the shorter piece is a **minor arc** and the longer one, a **major arc**. A **chord** of a circle is any line segment with endpoints on the circle. A **diameter** of a circle is any chord through the center of the circle, or the length of such a chord. A **circular region** is the set of points inside a circle. A **sector** of a circle is the region formed by two radii and the circle (a pie shape). A **segment** of a circle is the region formed by a chord of the circle and the circle. [21.1] Concentric circles are circles with the same center. [21.1] The **circumference** of a circle is its length. [23.2]

**comparison** language: Language that communicates either additive comparisons, as in “more/bigger/longer than” or “less than,” or multiplicative comparisons, as in “times as much as,” or both additive and multiplicative comparisons, as in “times bigger than” or “percent less than.” [20.1]

**composition** of rigid motions: the rigid motion that describes the net effect, original shape to final shape, when one rigid motion is followed by another. [22.4]
cone: the 3D surface resulting when a fixed point is joined to every point of a 2D curve, along with the region of the curve. The fixed point is the **vertex** (or apex) of the cone, and the 2D region is the **base** of the cone. In a **right circular** cone, the base is a circular region and the line through the vertex and the center of the circular base is perpendicular to every diameter of the base. In **oblique circular** cones, such a line is not perpendicular to every diameter. [21.2]

**congruent** shapes: shapes for which some rigid motion gives one shape as the image of the other; this rigid motion assures that the shapes are exactly the same shape and the same size. [22.1, 16.4 (3D)]

**conjecture**: a tentative result, usually based on one or more examples or drawings; an educated guess. [17.3]

**counterexample**: an example that shows that some statement is not true in general. [17.3 Learning Exercises]

cube: a **polyhedron** made up of 6 square regions; a regular **hexahedron**. [16.1, 16.5]

cylinder: the 3D surface formed by tracing around a 2D curve with a segment of a fixed length, so that the segment stays parallel to all its positions, plus the region of the 2D curve and the (parallel) 2D region defined by the other endpoint of the segment. The two regions are the **bases** of the cylinder. A **circular** cylinder results if the 2D curve is a circle, and is a **right circular** cylinder if the segment joining the centers of the two bases is perpendicular to every diameter of the bases, and an **oblique circular** cylinder otherwise. [21.2]

decagon: a **polygon** of 10 sides. [17.1]

degree (˚): (for angle measurement) the size of an angle formed by a ray \( \frac{1}{360} \) of a full turn from the other ray of the angle. Hence, there are 360 degrees in a full turn. A **minute** (‘) for angle size is \( \frac{1}{60} \) of a degree, and a **second** (") is \( \frac{1}{60} \) of a minute. [23.2]

diagonal of a polygon (prism): a line segment that joins two vertices of the **polygon** (prism) but is not a side of the polygon (…not completely in a face of the prism). [16.4 Learning Exercises (26.1 Learning Exercises)]

dodecahedron: a polyhedron with 12 faces; in a **regular dodecahedron** the faces are congruent regular pentagonal regions. [16.5]
edge of a polyhedron: see polyhedron. [16.2]; lateral edge: see prism and pyramid. [16.2] A hidden edge in a drawing of a 3D shape is an edge that cannot be seen from the viewpoint of the drawer; it is often indicated by dashed or lighter marks. [16.3]

English system of measurement units: a commonly used system in the United States, also called the British or customary system. A few of the units and their relationships follow.

<table>
<thead>
<tr>
<th>Length units</th>
<th>Area units</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 mile (mi) = 5280 feet (ft) = 1760 yards (yd)</td>
<td>Square inch (in²), square foot (ft²), and so on</td>
</tr>
<tr>
<td>1 yard = 3 feet</td>
<td>1 acre (A) = 43,560 square feet</td>
</tr>
<tr>
<td>1 foot = 12 inches (in.)</td>
<td></td>
</tr>
<tr>
<td>1 fathom (fath) = 6 feet</td>
<td></td>
</tr>
<tr>
<td>1 rod (rd) = 16.5 feet</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Volume units</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 gallon (gal) = 4 quarts (qt) = 8 pints (p or pt)</td>
</tr>
<tr>
<td>1 (U. S.) gallon (gal) = 231 in³</td>
</tr>
</tbody>
</table>

equianglar polygon: a polygon with angles all the same size. [17.1]

equilateral polygon: a polygon with sides all the same length. [17.1]

Euler’s formula for polyhedra: \( V + F = E + 2 \) (or any algebraic equivalent), where \( V \) is the number of vertices of a given polyhedron, \( F \) the number of faces, and \( E \) the number of edges. [16.2]

face of a polyhedron: see polyhedron; lateral face: see prism and pyramid. [16.2]

fixed point: see rigid motion. [22.3 Learning Exercises]

flip: an informal name for a reflection.

glide-reflection: a type of rigid motion, the composition of a translation and a reflection in a line parallel to the vector of the translation. [22.4]

great circle of a sphere: the largest circle possible on the surface of a sphere, with radius and center the same as those of the sphere. [21.2]

heptagon: a polygon of 7 sides. [17.1]

hexagon: a polygon of 6 sides. [17.1]

hexahedron: a polyhedron with 6 faces; a regular hexahedron is a cube. [16.5]
**hierarchy**: in our context, a classification system in which shapes in a sub-category have the properties of the category (as well as other properties not shared with all in the category). [17.2]

**hypotenuse** of a right triangle: the side opposite the right angle. [26.1]

**icosahedron**: a polyhedron with 20 faces; in a regular icosahedron the faces are equilateral-triangular regions. [16.5]

**image** of a point or shape: the corresponding point or shape that a transformation gives for the point or shape.[22.1] Finding images for a size change [20.1]; finding images for rigid motions [22.2]

**inductive reasoning**: using one or more examples as the basis for a **conjecture**. [17.2]

**isometric dot paper**: paper with dots arranged in an equilateral triangle pattern, useful for one type of drawing of 3D polyhedra. [16.2 Learning Exercises]

**isometry**: another name for a **rigid motion**. [22.1]

**kite**: a quadrilateral with two consecutive sides having the same length, and the other two sides also having the same length. [17.1, 17.2, 17.3]

**leg** of a right triangle: either of the two sides making the right angle. [26.1]

**length**: the characteristic of one-dimensional shapes that is measured with a ruler. Width, height, depth, thickness, perimeter, and circumference, for example, refer to the same characteristic. [23.2]

**line**: short for "straight line," a straight path that continues forever in two directions. A **line segment** is the piece of a line between two points called endpoints. A ray is the piece of a line that starts at a point and continues forever in one direction.

**measurement** of a characteristic: either the process of direct measurement, in which the given object is compared to a unit with respect the characteristic, or the resulting number and unit (for example, 3 inches). Standard units are most common, for purposes of communication and permanence (see, for example, **metric system** and SI). Key ideas of, [23.1]

**metric system**: the most common international system of standard units for measurement; see SI. The basic unit for length is the meter (metre); see **metric prefixes** [23.1, area units 24.1, volume units 24.2]
**net** for a 3D shape: a 2D pattern that gives a 3D shape when folded up. [16.1]

**n-gon**: a polygon of \( n \) sides, where \( n \) is a whole number greater than 2. [17.1]

**nonagon**: a polygon of 9 sides. [17.1]

**notations**: These are common and may even appear in elementary school mathematics textbooks. [17.1, 21.1]

Point: Capital letter (A, B,...)

Line segment with endpoints at P and Q: \( \overline{PQ} \)

Ray starting at C and going through D: \( \overrightarrow{CD} \)

Line through D and E: \( \overleftrightarrow{DE} \)

Polygon with vertices F, G, H, and I: \( FGHI \)

Line segments (and angles) may also be indicated with small letters, which may also mean their lengths (and angle sizes). A book may also use a darkened-in dot for an endpoint to show it is definitely included (and then the plain dot would mean the endpoint is not included).

Angles are named either by naming just the vertex, or if needed for clarity, as for angle D in the drawing above, naming a point on one side, the vertex, and then a point on the other side: \( \angle D \), or \( \angle CDE \)

Arcs of circles: \( \widehat{ST} \), with S and T the endpoints of the arc; \( \widehat{SUT} \) for clarity

**octagon**: a polygon of 8 sides. [17.1]

**orientation** of a figure: a clock direction (clockwise or counter-clockwise) assigned to a 2D shape. [22.2 Learning Exercises]

**parallel lines** (parallel planes): Lines in the same plane that never meet (planes that never meet). Notation: \( x \parallel y \). Compare **skew lines**.

**parallelogram**: a quadrilateral with both pairs of opposite sides parallel. To emphasize hierarchical concerns, **quadrilateral** may be replaced by **trapezoid**. [17.1, 17.3]

**pentagon**: a polygon of 5 sides. [17.1]

**pentomino**: a connected 2D shape made of 5 square regions with each square sharing at least one side with another square. [16.3 Learning Exercises]
**perimeter** of a polygon or closed curve: the distance along the polygon or closed curve. It is a type of length and must be kept distinct from the idea of area. [23.2]

**perpendicular bisector** of a segment: a line that passes through the midpoint of the segment and makes right angles with the segment. (construction of, 21.1)

**perpendicular** lines: lines that make right angles. [17.1, 18.2 Learning Exercises]

Perpendicular to a line (construction of, 21.1]. Notation: \(x \perp y\) means lines \(x\) and \(y\) are perpendicular

**pi** (\(\pi\)): the number that expresses the ratio of the circumference of any circle to its diameter. Pi does not have an exact terminating or repeating decimal so the small Greek letter \(\pi\) is used when the exact value is meant. Approximate values are \(\frac{22}{7}\) and 3.14, or more exactly but still approximately, 3.1415926536. [25.1]

**plane**: a perfectly flat, endless surface. A planar region is a region in a flat surface. [16, 17]

**polygon**: A closed plane figure made up of line segments joined end-to-end without crossing over. The line segments are called the sides of the polygon, and the endpoints are the vertices of the polygon (singular: vertex). A diagonal is a line segment joining two vertices that are not consecutive. [17.1] There are many special polygons. One way of classifying some of them is given below, in a hierarchy. [17.3] Shapes lower in the diagram are special versions of any shape that they are connected to higher in the diagram—for example, a rectangle is a special parallelogram, or a special quadrilateral. A hierarchy recognizes that any fact known about all parallelograms, say, applies to any special parallelogram like a rhombus or rectangle or square.
A **convex** polygon is a polygon for which every line segment joining points of the polygon lies entirely on or within the **polygonal region**; it is **concave** otherwise. [17.1 Learning Exercises]

**polygonal region**: All of the points “inside” a polygon. The polygon is just the line segments. The distinction is possibly important since teachers often use a cardboard cutout to illustrate a polygon. The edges of the cutout show the polygon, but the cardboard itself actually shows a polygonal region. [17.1]

![A triangle](image1)

![A triangular region](image2)

**polyhedron** (plural: **polyhedra** or polyhedrons): A closed 3D surface made up of planar regions (flat pieces). [16.2] The planar regions are the **faces** of the polyhedron. The line segments where faces meet are called the **edges** of the polyhedron. The points at which edges meet are called the **vertices** (singular: **vertex**) of the polyhedron. A diagonal of a polyhedron is any line segment that is not a side or a diagonal of a face but joins two vertices. [26.1 Learning Exercises] **Pyramids** and **prisms** are special polyhedra. If all the faces of a polyhedron **regular** polygons that are exactly alike and not arranged in some odd way, the polyhedron is called a **regular polyhedron**. [16.5]

**prefixes** used are of two types, general and metric:

<table>
<thead>
<tr>
<th>General (with some license)…</th>
<th>Latin</th>
<th>Greek</th>
<th>Latin</th>
<th>Greek</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>uni-</td>
<td>mono-</td>
<td>7</td>
<td>sept-</td>
</tr>
<tr>
<td>2</td>
<td>bi-</td>
<td>di-</td>
<td>8</td>
<td>oct-</td>
</tr>
<tr>
<td>3</td>
<td>tri-</td>
<td>tri-</td>
<td>9</td>
<td>non-</td>
</tr>
<tr>
<td>4</td>
<td>quad-</td>
<td>tetr-</td>
<td>10</td>
<td>dec-</td>
</tr>
<tr>
<td>5</td>
<td>quint-</td>
<td>pent-</td>
<td>100</td>
<td>cent-</td>
</tr>
<tr>
<td>6</td>
<td>sex-</td>
<td>hex-</td>
<td>1000</td>
<td>mill-</td>
</tr>
</tbody>
</table>

(The Greek prefixes for 10, 100, 1000, and so forth, are used in the metric system—for example, dekameter, hectometer, kilometer—whereas the Latin prefixes for 10, 100, 1000, and so forth are used for the subunits—for example, decimeter, centimeter, millimeter.) See the following table.
Metric system, with the last column applied to the length unit meter (m)

<table>
<thead>
<tr>
<th>Prefix</th>
<th>Symbol</th>
<th>Meaning of Prefix</th>
<th>Applied to Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>yotta-</td>
<td>Y</td>
<td>$10^{24}$</td>
<td>Ym</td>
</tr>
<tr>
<td>exa-</td>
<td>E</td>
<td>$10^{18}$</td>
<td>Em</td>
</tr>
<tr>
<td>peta-</td>
<td>P</td>
<td>$10^{15}$</td>
<td>Pm</td>
</tr>
<tr>
<td>tera-</td>
<td>T</td>
<td>$10^{12}$</td>
<td>Tm</td>
</tr>
<tr>
<td>giga-</td>
<td>G</td>
<td>1 000 000 000 or $10^9$</td>
<td>Gm</td>
</tr>
<tr>
<td>mega-</td>
<td>M</td>
<td>1 000 000 or $10^6$</td>
<td>Mm</td>
</tr>
<tr>
<td>kilo-</td>
<td>k</td>
<td>1000 or $10^3$</td>
<td>km</td>
</tr>
<tr>
<td>hecto-</td>
<td>h</td>
<td>100 or $10^2$</td>
<td>hm</td>
</tr>
<tr>
<td>deka-</td>
<td>da</td>
<td>10 or $10^1$</td>
<td>dam</td>
</tr>
<tr>
<td>no prefix</td>
<td></td>
<td>1 or $10^0$</td>
<td>m</td>
</tr>
<tr>
<td>deci-</td>
<td>d</td>
<td>0.1 or $10^{-1}$</td>
<td>dm</td>
</tr>
<tr>
<td>centi-</td>
<td>c</td>
<td>0.01 or $10^{-2}$</td>
<td>cm</td>
</tr>
<tr>
<td>milli-</td>
<td>m</td>
<td>0.001 or $10^{-3}$</td>
<td>mm</td>
</tr>
<tr>
<td>micro-</td>
<td>$\mu$</td>
<td>0.000001 or $10^{-6}$</td>
<td>$\mu$m</td>
</tr>
<tr>
<td>nano-</td>
<td>n</td>
<td>0.0000000001 or $10^{-9}$</td>
<td>nm</td>
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<td>p</td>
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<td>zm</td>
</tr>
<tr>
<td>yocto-</td>
<td>y</td>
<td>$10^{-24}$</td>
<td>ym</td>
</tr>
</tbody>
</table>

**prism**: A polyhedron having two faces (called **bases**) that are parallel and congruent (informally, “exactly alike in size and shape”) and whose other faces (called **lateral faces**) are parallelogram regions (or special parallelogram regions such as rectangular ones) formed by joining corresponding vertices of the bases. Some edges of a prism are on the two bases; the other edges are called **lateral edges**. [16.2] A prism is a **right prism** if the lateral edges (the ones not on the bases) are perpendicular to the edges at the bases, and **oblique** otherwise. [16.2 Learning Exercises]
**protractor**: a tool for measuring angles. [17.1, Appendix F]

**pyramid**: A polyhedron with one face being any sort of polygonal region (often called the **base** of the pyramid) and with all the other faces being triangular regions with one vertex in common. 16.2 The edges not on the base are called **lateral edges**, and the non-base faces are called **lateral faces**. If the base is shaped like a regular polygon and the lateral faces are all congruent, the pyramid is called a **regular pyramid** (your shapes A and G). [16.2 Learning Exercises]

![An oblique pentagonal pyramid](image1)
![The base of the pyramid (shaded)](image2)
![The lateral edges of the pyramid highlighted](image3)

**Pythagorean theorem**: the important relationship among the lengths of the three sides of a right triangle: The sum of the squares of the lengths of the **legs** equals the square of the length of the **hypotenuse**. [26.1]

**Pythagorean triple**: three non-zero whole numbers, $a$, $b$, and $c$, related by their squares as in the Pythagorean theorem: $a^2 + b^2 = c^2$. [26.1 Learning Exercises]

**quadrilateral**: a polygon of 4 sides. [17.1]

**rate**: a ratio that stays the same as the quantities involved vary. [26.2]

**ray**: see **line**.

**rectangle**: a parallelogram with a right angle. (This definition necessarily leads to all four angles being right angles.) [17.1, 17.3]

**reflection** in a line: a 2D **rigid motion** in which a point on either side of the line has its image as though the line were a mirror. See also **symmetry**. [22.1, 22.2]

**reflection symmetry** with respect to a plane, for a 3D shape: a reflection in a plane that gives the original shape as the image. [18.2]

**regular polygon**: a polygon that is both **equiangular** and **equilateral**. [17.1]
**regular polyhedron**: a polyhedron with faces all the same regular-polygonal regions and with the same arrangement at each vertex. Also called **Platonic solids**. [16.5]

**representation**: a way of communicating or thinking about something; in geometry, any of a drawing, a model, a net, a word, or even an equation may be a representation. [16.3]

**rhombus**: a **kite** that is also a **parallelogram**. (This definition necessarily leads to all four sides having the same length.) [17.1, 17.3]

**rigid motion**: a matching of the points in the plane (or space) with points, so that original lengths are the same as lengths in the **images**. [24] A **fixed point** for a rigid motion is a point that is its own image. [22.3 Learning Exercises]

**rotation** about a point: a 2D rigid motion in which the plane is “turned” about a point, called the **center** of the rotation. See also **symmetry**. [22.1]

**rotational symmetry** with respect to a point, for a 2D shape (or a line, for a 3D shape): a rotation about the point, called the **center** (or line, called the **axis**), such that the image is the same as the original shape. [18.1 (18.2)]

**scale factor**: the common ratio of image length to original length, for a **size change**. [20.1 (2D), 20.3 (3D)]

**sector** of a circle: see **circle**. [21.1]

**segment** of a circle: see **circle**. [21.1]

**SI**: The metric system of standard units. SI is short for **Système International d'Unités**, the International System of Units. The strengths of the system are that subunits and larger units are related to a basic unit by a power of 10 and that the basic units are carefully defined for reproducibility (in most cases). The table for metric prefixes (under **prefixes**) gives the subunits and larger units and in that table are applied to the meter (symbol: m), the basic unit for length. [23.1]

**similar shapes**: shapes that are related by a size change, possibly along with a rigid motion of some sort. Noun: **similarity**. [20.1 (2D), 20.3 (3D)]

**size change** or **size transformation**: a matching of the points of a plane (or space) such that the size of every angle is the same as in its image and such that the ratio of an image length to
the original length is always the same value, called the **scale factor**. The **center** of a size change is the point with which the matching is done. [20.1]

**skew lines**: lines in space that never meet but are not parallel (for example, the bottom edge of front wall, and the top edge of side wall). Contrast **parallel lines**, which are in the same plane.

**slide**: an informal name for a **translation**.

**sphere**: the 3D set of points at a fixed distance from a fixed point, called the **center** of the sphere. 21.2 A **hemisphere** is half a sphere. [21.2 Learning Exercises]

**square**: a rectangle that is also a rhombus. (This definition necessarily leads to all four angles being right angles and all four sides having the same length.) [17.1, 17.3]

**standard units**: in measurement, an accepted system of units, like the **metric system**. [23.1]

**surface area**: see **area**. 24.1

**symmetry** of a (2D or 3D) shape: a rigid motion that gives an image that is the same shape as the original shape. The type of rigid motion may be suggested by an adjective: reflection symmetry or line symmetry or mirror-image symmetry; or rotational symmetry. [18.1, 18.2]

**tangent** to a circle: a line that has exactly one point in common with the circle. A tangent is perpendicular to the radius to that point. [21.1 Learning Exercises]

**tangram**: a type of puzzle, typically made up of pieces cut in a certain way from a square region. [17.1 Learning Exercises]

**tessellation** of a plane (space): a covering of a plane (space) with a limited number of types of regions. In a **regular** tessellation of the plane, all the regions are one type of regular polygonal regions. [19.1 (2D), 19.2 (3D)]

**tetrahedron**: a polyhedron with 4 faces; a **regular** tetrahedron has faces that are equilateral-triangular regions. [16.5]

**tetromino**: a connected 2D shape made of 4 square regions with each square sharing at least one side with another square. [16.3 Learning Exercises]

**transformation geometry**: a term for school geometry that features **rigid motions** and **size changes**. [22]
**translation**: a type of rigid motion in which the image of a point is the point a fixed distance in a fixed direction from the original point. [22.1]

**trapezoid**: a quadrilateral with at least one pair of opposite sides parallel. Many books use “exactly one pair,” but the definition here allows the trapezoid family to relate to other special quadrilaterals. To retain that value without causing other difficulties, the **isosceles trapezoid** is defined awkwardly, as a trapezoid in which both angles adjacent to one of the parallel sides are the same size. Under either definition, the two sides of an isosceles trapezoid that may not be parallel have the same length. [17.1 Learning Exercises, 17.3]

**triangle**: a **polygon** of three sides. In an **acute triangle**, all three angles are acute; a **right triangle** has a right angle, and an **obtuse triangle** has an obtuse angle. The lengths of the sides of a **scalene triangle** are all different, but an **isosceles triangle** has at least two sides with the same length, and an **equilateral triangle** has the same length for all three sides. See also **polygon**. [17.1]

**triangular numbers**: the numbers 1, 3, 6, 10, 15, ..., \(\frac{n(n+1)}{2}\), ..., so called because they give counts of dots arranged in a triangle, like bowling pins for the fourth triangular number \((n = 4)\). [17.1 Learning Exercises]

**turn**: an informal name for **rotation**.

**unit**: in measurements, the size of the comparison object, for example, mile; see **standard units**. (With fractions, what = 1.) [23.1]

**vector**: the direction and distance associated with a given **translation**, often shown by an arrow of the correct length and pointed in the correct direction; in general, a quantity that may have two or more sub-quantities. [22.1]

**vertex** of a polygon: see **polygon**; of a polyhedron, see **polyhedron**; of an angle: see **angle**.

**volume** of a closed 3D region: The number of 3D units that could fit inside the region, or that could be used to make an exact model of the region. The units are usually cubic regions (for example, cm\(^3\)). The metric system also uses the liter (litre). [24.2, formulas for determining volumes, 25]
Answers and Hints for Exercises (Instructor’s Version)

16.1 Shoeboxes Have Faces and Nets!
1. Your “box” should be taller than it is wide or deep.

2. Did you start at the top and take advantage of the last cube drawn when adding later cubes?

3. a. 5 faces, 5 vertices, 8 edges       b. 5 faces, 6 vertices, 9 edges

4. The net shows only 5 faces, and a cube has 6 faces.

5. a. 

   Front view
   
   Right side

   Top view (from front)

   b. Do you see two possible answers for the top view, depending on what might or might not be hidden in the back left corner?

   c. They are like views from the reverse (or mirror images).

6.

The shape in the example (Ex. 5)

5. a (one possibility)

5. b (one possibility)

16.2 Introduction to Polyhedra
1. a. B, E (squares are special parallelograms), F

   b. G

   c. A, C, H (and M, if used)

   d. D, G, I, J

   e. H, J, K (and L and M, if used)

   f. K (What about C, E, F, and J? Some people regard a rectangle as a special isosceles trapezoid, as we will see later.)

   g. A, E, H (and L and M, if used)
2. a. Lateral edges of a prism are equal in length and parallel to each other. The lateral faces of a prism are parallelogram regions (or special parallelogram regions, such as rectangular regions).
   b. 50 edges, one edge from each vertex of the base to the vertex not on the base; 50 faces, one face for each side of the base.

3. a. A hexagonal pyramid

4. There is a polyhedron with the fewest number of vertices: a triangular pyramid (4 vertices); if there were only 3 vertices, they would all lie in the same flat surface (plane) and not give a polyhedron. But a polyhedron may have any large number of edges; consider a prism or pyramid with a base with a large number of vertices. So there is not a polyhedron with the greatest number of vertices.

5. a. Unless you have a weird polyhedron of some sort that is not in the kit, the relationship will hold.
   b. 7
   c. 12
   d. No, because... (Instructor: \( F + V \) would then be even, but \( E + 2 \) would be odd.)

6. a. right rectangular prism
   b. right hexagonal prism (the common kind of pencil)
   c. right rectangular prism
   d. There are different possibilities, including right rectangular prism and parallelogram right prism for the most common ones.

7. a. The 4 edges on the base were counted twice, as were the 4 lateral edges.
   b. Each vertex is on 3 faces, so it is counted 3 times in the argument that looks at the vertices on the 6 faces separately.

8. (Instructor: a. For an \( n \)-gonal pyramid, there are \( 3n \) angles at the \( n \) vertices on the base, plus \( n \) angles at the other vertex, for a total of \( 4n \) angles.
   b. Similarly, with 3 angles at each of the \( 2n \) vertices, there are \( 6n \) angles for an \( n \)-gonal prism.)

9. a. C, E, F
   b. rectangular regions
   c. A and G

10. a. 14 (12 usually, but if the bases are….)
b. 24 (16 usually, but if the bases are….)

11. Be sure to think about measurements, not just counts.

### 16.3 Representing and Visualizing Polyhedra

1. a. Looking straight at the midpoint of an edge with the cube turned so the same amount seems to be above the edge as below the edge.
   b. Looking from above and in front of an upper-right vertex from the right.
   c. Looking straight on at one face of the cube.

2. a. Maximum number of faces: 3; minimum: 1
   b. Maximum number of vertices: 7
   c. Maximum number of edges: 9

3. **Hint:** Add three hidden edges to the base.

4. a. **Hint:** See Figure 1.
   b. **Hint:** Be sure that your lateral edges appear to be parallel and perpendicular to the edges of the base that they meet.
   c. **Hint:** Adjust the quadrilateral pyramid in Figure 8.
   d. **Hint:** Sketch the top base first, remembering that you are to look at the prism obliquely. Then sketch the lateral edges, keeping them parallel and of the same length. Finally, put in the edges of the bottom base, keeping in mind which edges are hidden.
   e. **Hint:** See Learning Exercise 3.
   f. **Hint:** See Figure 7.
   g. **Hint:** Learning Exercise 10 in Section 16.2.
   h. One possibility is a quadrilateral pyramid. Why isn’t it possible with a prism?

5. a. 1-6, 2-4, 3-5
   b. 1-3, 4-6, 2-5
   c. 3-5, 1-4, 2-6
   d. 2-4, 3-6, 1-5

7. Nets a and c only (Did you notice that b and d are really the same net?)

8. The view is from above and in front of but to the right. Although equal or parallel lengths in the actual polyhedron will also be equal or parallel in the isometric sketch, right angles in the actual polyhedron are not right angles in the isometric drawing (they only appear to be, because our minds can process the drawing).
   b. yes
   c. right hexagonal prism
e. Volume: 4 cube regions; surface area: 18 square regions

9.

a. ![Diagram]

b. ![Diagram]

c. ![Diagram]

There might be other possibilities; compare yours with someone else’s. This task is often difficult—what strategies did you use (for example, focusing on edges that are not cut? Holding a cube as you imagined the cuts? Turning the net to make it easier to visualize where the base is? Working backwards from a cut-out net to the cube?).

10.

a. ![Diagram]

b. ![Diagram]

c. Same as part (b). (Do you see why?)

In each case, your version might be turned or even flipped from the answer here.

12. a. **Hint**: There are 5 differently shaped tetrominoes.

   Instructor:

   ![Tetrominoes]

   b. There are 12 pentominoes. Did you find them all? How can you be sure?
Instructor (the darker squares may suggest how to build on the tetrominoes):


Each has area $= 5$ square regions, but the perimeters…

d. No, for example... (Instructor: Use some of the pentominoes.)

e. No, for example... (Try some rectangular regions with the same perimeters.)

(Instructor: For example, a 2-by-4 rectangular region and a 1-by-5 one.)

13.

14. a. a rectangular region, not square
   b. a rhombus region

15. There are different possibilities for each part. One somewhat laborious way to check is to cut out your net and fold it. Another (recommended) way is to have a classmate or two look at yours.

16. a. triangular right prism
   b. cube
17. a.  

![Diagram of a 3D shape]

b.  

![Diagram of another 3D shape]

18. a. FE, AD, AB, FH  
b. BC, GH, and FE

19. a. Instructor  

![Diagram of a 2D shape]

b.  

![Diagram of another 2D shape]

c. Instructor

![Diagram of a 3D shape]

16.4 Congruent Polyhedra

2. No—none can be moved to match another exactly.

3. The hidden shape will have the same sized faces and (as a result) the same areas as the given shape, so 108 cm².

4. a. Yes. A rotation of 360° (or 0°) will make it match itself.  
b. Yes. Just reverse the motions that showed that P was congruent to Q.  
c. Yes. Do all the motions that show that R is congruent to S and then continue with the motions that show S is congruent to T.

5. Instructor: There are the 5 suggested by the 5 2D tetrominoes (see below), plus another couple pictured below to the right (for us, the mirror image of the next-to-last one would be considered to be congruent).

![Diagram of tetrominoes]

7. a.  

![Diagram of a polyhedron]
9. Possibility: Two polygons are congruent if one polygon can be moved (by rotating, reflecting, or sliding, or a combination of such motions) so that it would fit on the other polygon exactly.

16.5 Some Special Polyhedra

1. a. AT  b. ST  c. ST  d. AT  e. ST  f. ST  g. AT  h. ST  i. ST  j. ST  k. ST
   1. AT

2. Knowing how many types (and what the types are) can enable you to ask, “Why are these the only ones?” and perhaps reveal something important about them. If only regular polyhedra can appear in some context for some theoretical reason, then you will know ahead of time what the possibilities are.

3. a. A 20-faced polyhedron (icosahedron), perhaps regular, made up of 20 triangular regions
   b. A 12-faced polyhedron (dodecahedron), perhaps regular, made up of 12 pentagonal regions
   c. An 8-faced polyhedron (octahedron), perhaps regular, made up of 8 triangular regions.

4. a. Check to see whether there is the same arrangement of faces at every vertex.
   b. Try any quadrilateral prism that is not a cube.
   c. Some of the triangles in your net should not be equilateral triangles.

5. a. The tetrahedron may be more easily seen with an actual cube, especially if the cube is transparent.
   b. The other four vertices also give a regular tetrahedron.

6. Euler’s formula, $V + F = E + 2$, does hold. It is curious that some of the same numbers occur with the cube-octahedron and the dodecahedron-icosahedron.

7. a. First polyhedron: Equilateral triangles, regular hexagons; second polyhedron:
   - Squares, regular hexagons
   b. Euler’s formula (see answer to Exercise 6) does hold, as it does for both polyhedra, although without the hidden edges in the second, it is likely impossible to count $V, F,$ and $E$.
   c. Instructor: A natural question is, “How many types of semiregular polyhedra are there?”
9. a. Yes, because adjacent faces share edges, and all the edges of a given face are the same length.
b. Yes, because each face is the same type of regular polygonal region.
c. Yes, for the same reason as for part (a).
d. No. The regular polygons may be different types, as a square and a hexagon.

16.6 Issues for Learning: Dealing with 3D Shapes

1. a. They might have counted the visible square regions.
b. One possibility: Double the count as in part (a), for the hidden parts. Or, count the 4 in front, double for the back, giving 8. Do the same for the right and left sides, for another 8. Finally, get another 8 for the top 4, plus 4 for the bottom.
c. The counts for the right and left are counting cubes that have been counted already.
2. a. For example, the 9 in the front and the 12 on the right both count the 3 cubes at the front right corner.
b. The student is overlooking the cubes in the inner, middle columns.

17.1 Review of Polygon Vocabulary

1. a. scalene right triangle
b. trapezoid
c. equilateral triangle (or regular triangle)
d. rhomboidal region
e. regular pentagon
f. parallelogram
g. square
h. (equiangular) hexagon
i. equilateral 12-gon, or concave equilateral 12-gon (see Exercise 6).
   Using less-well known prefixes, it is also an equilateral dodecagon.
j. rhombus
2. (Instructor: Only parts (g), (l), and (q) are impossible. Ask why.)
3. (Instructor only)
   a. Each vertex is on two sides, so the way of counting given will count each vertex twice.
b. Each side actually makes two angles, one at each endpoint; the way of counting given does not take that fact into account.
4. a. a regular hexagonal region and isosceles triangular regions
b. rectangular regions
c. a rectangular region and triangular regions, probably isosceles

d. rectangular regions and isosceles trapezoidal regions

5.  a. Most of them are isosceles right triangles, but there are also a square and a parallelogram.

b. There are 7 isosceles right triangles, 2 (overlapping) isosceles trapezoids (not counting special ones), 3 non-isosceles trapezoids (not counting special ones), and 1 parallelogram region (not counting special ones).

c. (Instructor only) The different sizes are 1/4 (two pieces), 1/8 (three pieces), and 1/16 (two pieces). Rather than trust one’s eyes, copying and cutting out the pieces, and comparing them, gives a convincing justification. You might also justify the results by using area formulas.

6. (Instructor only)

b. Informally, one might say, “A polygon that does not have any dents,” or “One that does not pooch in.” A common technical definition is “A polygon is convex if, whenever two of its points are joined, the line segment never goes outside the polygonal region.” The idea extends to nonpolygonal shapes.

d. A kite is a quadrilateral that has two consecutive sides the same length, with the other two sides also having equal lengths.

7. (Instructor only)  a. The "fat" rhombuses in Pattern Blocks; join the center of the hexagon to every other vertex of the hexagon.

b. The equilateral triangles in Pattern Blocks; join the center of the hexagon to each vertex of the hexagon.

c. The isosceles trapezoids in Pattern Blocks; cut the hexagon through its center.

8. Polygon: Number of sides, number of angles, length of each side, size of each angle, total of the lengths of the sides (the perimeter), total of the sizes of the angles, and so on. (See Learning Exercise 9.) Polygonal region: Same as polygon, plus the area of the region.

9. A table helps to see a pattern that may be less obvious if the results are just written, as with 5 vertices, 5 diagonals; 6 vertices, 9 diagonals, and so on. The pattern can often then be extended from one line to the next (for a 12-gon, 54 diagonals; for a 20-gon, 170...
diagonals), but without seeing the general result for \( n \) vertices. Possible (and mysterious) 

**Hint 1:** Add a column for twice the number of diagonals—can you relate that column to the column for the number of vertices? A pattern suggests an educated guess, but, as you will see, patterns cannot always be trusted! Hence, now that you have a conjecture, try to reason why it must be true. Better Hint, because it gives a general argument: Each vertex in an \( n \)-gon is joined to all but 3 of the vertices to give \( n - 3 \) diagonals at each vertex, but doing this at each of the \( n \) vertices will count each diagonal twice.

(Instructor: For an \( n \)-gon, there will be \( \frac{n(n-3)}{2} \) diagonals.)

10. The sum of the sizes of the angles in each of the triangles is 180°. When you add all of those up, you are also adding the sizes the angles of the quadrilateral.

11. (Instructor: For the 20-gon, \( (20 - 2)\times 180° = 3240° \). In general, the “angle sum” (the sum of the sizes of the angles) will be \( (n - 2)\times 180° \). Ask whether there is a justification beyond seeing a pattern.)

12. a. 95°  b. 102°  c. 360 - (85 + 78 + 90) = 107°  d. 73°  e. 90°  
   f. 130°  g. 360 - (130 + 109 + 52) = 69°  h. 111°  i. 71°  j. 128°  
   k. The sum of the sizes of the exterior angles, one at each vertex, will be 360° for a quadrilateral. (Parts (k) and (l) Instructor only)

13. a. \( \frac{(5-2)\times 180}{5} = 108° \)  b. 72°  c. 135°  d. 45°  e. \( \frac{(n-2)\times 180}{n} \)  
   f. 180 - \( \frac{(n-2)\times 180}{n} \) = \( \frac{360}{n} \) (13(f) Instructor only)

14. a. 90°  b. 67°  c. 61°  d. 84°  e. All five angle sizes should add to \( (5 - 2)\times 180 = 540° \), so 540 = 90 + 113 + 119 + 96 + e, or \( e = 122° \)  f. 58°  
   g. 72°  h. 59°  i. 52°  j. 101°  k. 79°  m. 62°  
   n. The sum of the exterior angles, one at a vertex, for a pentagon is 360°.  
   o. \( 5\times 180 - (5 - 2)\times 180 = 360 \)  (14(n) and (o) Instructor only)

15. Each is true. Your reasoning? (Instructor: 15(a). The right angle uses 90° of the 180° for all three angles, so the sizes of the other two angles must be 90. That is, the acute angles are complementary. 15(b). If a triangle had more than one right angle, then sum of the sizes of its angles would be more than 180°. Students may not be comfortable with the indirect reasoning in 15(b).  15(c). \( x + (180 - x) = 180 \)
16. b. \( n^2, (x+1)^2 \)
   c. **Hint:** It may be helpful to add a new column, 2 \( x \# \) dots and see how the entries in that column are related to the number in the first column, or to consider the differences between successive numbers of dots. Once you have a conjecture based on a pattern, see whether you can give an argument that justifies the conjecture. **Instructor:** Making an \( n \)-by-\( (n+1) \) “rectangle” with two copies of the right-triangle version helps to see the \( \frac{n(n+1)}{2} \) result for the \( n \)th triangular number.

18. (Instructor only)
   a. rectangular region
   b. triangle
   c. triangular region
   d. parallelogram region for the lateral faces, polygonal region of some sort for the base
   e. triangular regions for the lateral faces, polygonal region for the base
   f. rectangle, most commonly
   g. hexagon
   h. hexagonal region

19. Your trials should support the conjecture, because it is true in general.

20. (The sides of \( \angle IBA \) can be rays.)

21. There are many possibilities. Check to make sure the rays start at the correct points, and that D is the vertex of angle FDE. TUV should be shaded in, but not PQRS.

### 17.2 Organizing Shapes

1. The classification has each type of quadrilateral in a separate category, even though they share many characteristics besides having four sides. (The circles are a common way of showing the categories, and are not to be viewed as quadrilaterals themselves.)

2. a. AT   b. ST (Sketch a nonsquare rectangle.)  c. ST (Sketch a parallelogram that has no right angles.)  d. AT   e. ST  f. ST  g. ST

   (Instructors only: h. AT i. AT j. ST  k. ST  l. ST  m. ST  n. ST)
3. Samples:  a. Shared: opposite sides parallel and equal in length; opposite angles the same size, and so on. Different: Possible for angle sizes to differ, lengths of diagonal can differ, and so on.

b. Shared: Have pairs of adjacent sides the same length, diagonals make right angles, pairs of opposite angles have the same size, and so on. Different: Possible for angle sizes to differ, possible for some sides to be different lengths, and so on.

c. Shared: have 4 sides, 4 angles, 2 diagonals, angle sum is 360°, and so on. Different: Trapezoids have parallel sides, quadrilaterals may not; and so on.

d. Shared: Two sides parallel, four sides. Different: Other pair of sides may not be parallel, sides (or angles) may all be different sizes, and so on.

e. Shared: Both pairs of opposite sides parallel and equal in length, opposite angles are the same size, diagonals bisect each other, and so on. Different: Sides may not all have the same length, diagonals may not be perpendicular, and so on.

4. Three are not possible. (Instructor: Parts (c), (e), and (f) are not possible.)

5. One possibility, using two new categories...

(closed shapes) (not closed shapes)

- ovals
- ellipses
- circles

- zig-zags
- angles

- squiggles
- parabolas

6. a. That the parallel sides are horizontal, with one shorter than the other and above it.

   b(i). Perhaps the student thinks a parallelogram's angles cannot be right angles—that a parallelogram's sides must be “tilted.”

   b(ii). Perhaps the student thinks a rectangle must have unequal dimensions.

   (iii, iv Instructors only)

   b(iii). Perhaps the student thinks that the angles of a rhombus cannot be equal in size.

   b(iv). Perhaps the student thinks that the sides of a kite cannot all be the same length.

17.3 Triangles and Quadrilaterals

1. Kites and isosceles trapezoids do not fit easily into this Venn diagram.
2. (Instructor only) “Opposite sides/angles are equal” means both pairs. For the conjectures listed, there are counterexamples for each conjecture for quadrilaterals and trapezoids and kites, for all but the 3rd conjecture for isosceles trapezoids, for the 3rd and 4th for parallelograms, for the 4th for rectangles. All the conjectures are true for squares, and all but the 3rd is true for rhombuses. Following is a summary compiled with one class; the conjectures are numbered and only the number is given when the conjecture is automatically true from the hierarchy.

3. (Instructor only) In each case, the properties should apply to the more special polygon.
4. Stuck? Measure segments and angles—are there any possible relationships?
   Instructor:  a. The segment joining the midpoints is parallel to, and half the length, of the
top third side.  b. The part (a) results hold, of course, and the four triangles are “exactly the
same shape” (congruent), hence each has one-fourth the area of the large triangle.

5. a. The angles can be placed next to each other so that the two outside sides appear to lie
along a straight line, and their sum is therefore 180°.
b. The new “placements” of the three angles again appear to lie along a straight line. The
method does work with obtuse and right triangles, by folding down the vertex with the
largest angle size.

6. (Instructor only)
a. \(x = 72°; y = 142°\)
b. \(n = 112°; p = 36°; q = 144°\)
c. \(s = 75°; r = 95°\) (using the fact that the sum of the sizes of the four angles of the
quadrilateral is 360°)

7. a. After the last clue, the shape must be a parallelogram.

8. a. Pat could have been thinking \(a - \frac{a}{a+1} = a \times \frac{b}{c}\) or perhaps \(a - \frac{a}{a+1} = a \times \frac{a}{a+1}\) (or perhaps
something else).
b. The first conjecture is not true in general; find a counterexample. But the second one
is correct; try different values for \(a\), and they will strengthen your belief in it. Use
your algebra knowledge to show that \(a - \frac{a}{a+1}\) is indeed always equal to \(a \times \frac{a}{a+1}\).
   (Instructor: A similar situation can be built on an example or two suggesting correctly
that \(a + \frac{a}{a-1}\) is also equal to the product, \(a \times \frac{a}{a-1}\), but an attractive \(a + \frac{a}{a-1}\)
conjecture is not true in general. The problems are from Usiskin.)

9. The diagram will look like the one for quadrilaterals, but with “prisms” attached to each
type of quadrilateral.

10. Think hierarchically (every rhombus is a special trapezoid, so the result should be the
same). Or, each of the faces in each case is a quadrilateral, so the sum of all the angles in
all the faces will be the same for any quadrilateral prism, 2160°.

17.4 Issues for Learning: Some Research on 2D Shapes

1. Relevant: 4 sides, 2 sides parallel but not necessarily the “top” and “bottom.” Your
drawings should include the parallel sides not at the top and bottom, top longer than
bottom, 2 or 3 sides equal in length, a very small one, one with 2 right angles, and one
not “cut off” (perhaps like the one with 2 right angles, but with the right angles at the top).

2. It is likely that the examples that the child had seen did not have any angles small in size.

3. The child likely is focusing on the lengths of the visible parts of the sides of the angle.

4. Does your collection include rhombuses with 1, or 2, or 4 right angles and with different orientations of the parallel sides?

5. a. Except for the choice of 6, the child seems to be looking for right angles at all the vertices.
   b. Perhaps the child is looking for four sides of about the same length, for square, and shapes evenly “stretched out” for rectangle.

### 18.1 Symmetry of Shapes in a Plane

1. Having reflection symmetry: \(\text{ABCDEH(2)I(2)MO(2)TUVWX(2)Y}\)
   Having (non-trivial) rotational symmetry: \(\text{HINOSXZ}\)

2. Instructor: For example, for reflection symmetry: some leaves of trees or blades of grass or flower blossoms, some shells that are nearly flat,…; for rotational symmetry: the moon as it appears to us, some flower blossoms,…

3. Instructor: For example, for reflection symmetry: a clock face (without numbers), a door (ignoring knob and hinges), most walls,…; for rotational symmetry: a pizza pan, the clock face and door again,…

4. a. 1 reflection symmetry only (Don't count a 360° rotation unless there are other rotational symmetries.)
   b. 2 reflection symmetries; 2 rotational symmetries
   c. 2 rotational symmetries (180° and 360°)
   d. 1 reflection symmetry
   e. 0 symmetries
   f. 2 reflection symmetries; 2 rotational symmetries
   g. 1 reflection symmetry

5. a. The bisector of the angle.
   b. Don't overlook the lines themselves.
   c. There are actually infinitely many — do you see that?
   d. Any line through the center of the circle is a line of symmetry for the circle, so there are infinitely many lines of symmetry for a circle.
e. 2 reflection symmetries  (Do you also see 2 rotational symmetries?)

6. A rotation of *any* number of degrees, say 3.6° or $4\frac{2}{3}$°, will be a rotational symmetry.

7. (Instructor only)

7. Samples

a.  

b.  

c.  

d.  

e.  

8. (Instructor only)

a. 1 reflection symmetry

b. 1 reflection symmetry

c. 2 rotational symmetries

d. 1 reflection symmetry

e. 20 rotational symmetries (18°, 36°, and so on)

f. 1 reflection symmetry

10. a. The segments must have the same length, so the triangle must be isosceles.

b. The two angles must have the same size.

c. M must be the midpoint of segment BC.

d. The two angles must have the same size.

11. (Instructor only)  a. They must have the same length, because one would fit on the other exactly if the dashed line is a line of symmetry.

b. They must have the same length.

c. Nothing relates the lengths of these two segments.

d. These two angles must have the same size, because one would fit on the other exactly if the dashed line is a line of symmetry.

e. Because A would fit on B, the dashed line goes through the midpoint of segment AB. Similarly, for segment ED. The angles at E and D will have the same size, as will the angles at A and B.

12. Segment GH will have the same size as segment KJ, as will segments LG and JI, as well as segments LK and HI. Angles G and J will have the same size, as will angles K and H, and angles L and I.

13. (Instructor only) a. The reflection symmetry shows that the two diagonals would fit on each other, so they must have the same length. (Notice that this is not true for ordinary trapezoids.)
b. The result should apply to rectangles because they can be regarded as special isosceles trapezoids. It is not true for parallelograms.

b. The result should apply to special parallelograms, but it is not true for all kites.

15. a-b. Use the "long" diagonal as a line of symmetry.
c. Use the 180° rotational symmetry of a rhombus (or a possible fact about parallelograms and the hierarchy), and a reflection symmetry.
d. Use two reflection symmetries. (Instructor: There may be other arguments.)

### 18.2 Symmetry of Polyhedra

1. No, it is not possible. This right-versus left-handedness requires a reflection.

2. (Instructor only) a. 6 different planes of symmetry, one through each edge.
b. 1

c. Only 3 (why don't the “other” inviting planes work?)
d. 5

3. (Instructor only) Remember that for a given axis, there may be more than one rotational symmetry possible, using different numbers of degrees.

a. 4 axes of rotational symmetry, each has a 120° and 240° rotation, plus the 360° rotation that gives the same result for each axis, so there are 9 total rotational symmetries for Shape A. (Instructor: That the 360° rotation counts just once may need discussion; as you know, the different descriptions entail the same beginning-end results, so they are describing just one function from the mathematical view.)

b. 0 axes of symmetry (the trivial 360° rotation is not usually counted, unless there are other rotational symmetries)

c. Each of the 3 axes of rotational symmetry can involve a 180° rotation, or the 360° one. The latter would give the same result for each axis, so it is counted just once.

d. 1 axis of symmetry, with 72°, 144°, 216°, 288°, 360° rotations

4. There are 3 through pairs of opposite edges, plus 6 planes (through vertex, perpendicular to edge) for reflection symmetries, and 3 axes of rotational symmetry, each allowing 90°, 180°, and 270° rotations (plus the 360° one), plus 4 axes allowing 120° and 240° rotations (plus the 360° one), plus 6 axes allowing only 180° (and 360°) rotations. Amazing!

5. a. There are 2 planes giving reflection symmetries; they are.... There is 1 (nontrivial) rotational symmetry of 180° (what is the axis?).
b. No symmetries exist.

c. There are 2 planes giving reflection symmetries and 2 rotational symmetries, counting the 360° rotational symmetry.

6. There are different possibilities. Compare with others. Do any require fewer than twice the original number of cubes?

7. Hint: The base must be a square.

8. 9 reflection symmetries and 16 rotational symmetries (8 from one axis, and 2 for 8 other axes, but each of the latter counts the 360° rotational symmetry counted with the first axis)


10. Does each have the same final effect, regardless of any difference in motion along the way?

11. Regular octahedron (shape H): 9 planes of reflection symmetry; 13 axes of rotational symmetry (giving 36 different rotational symmetries!)

   Regular dodecahedron (shape L) and regular icosahedron (shape M): 15 planes of reflection symmetry; 31 axes of rotational symmetry (giving 90 rotational symmetries!).

19.1 Tessellating the Plane

1. What causes the difficulty?

4. If you have trouble, rotate the shape 180° about the midpoint of each side.

5. Each type of Pattern Block will tessellate the plane.

6. Examples

   a.

   b.

7. After experimenting a bit, you will believe, Yes.

8. c. The area of the larger triangle is 4 times the area of the smaller one. (Note that the lengths of the sides of the larger one are only twice the lengths of the sides of the smaller.)
9. (Instructor only) Each has many rotational symmetries (different centers) and translation symmetries (different distances). Translations come up in Ch. 22, so students' language may be different. Parts (b) and (c) have many reflection symmetries (different lines).

19.2 Tessellating Space

1. Yes, there are many.... (Instructor: Abutting two bricks at right angles at the end of one brick, making an L shape, is fairly common. Sliding one brick along another, but not halfway, gives lots of other possibilities.)

2. b, d

3. a, d, e; in (a), the curved spine could fill in the curved-in space, but in (b) the shape could not fill in the curved space. In (c), the piece(s) will not fit the inward-going spaces.

4. Volume is most often based on the number of cubical regions that fill a space; this space-filling by cubical regions is a tessellation.

5. Instructor: Three pyramids I can be arranged to make a cube, so shape I can tessellate space.

6. Instructor: Neither J nor K will tessellate space (even though it looks as though they should).

20.1 Size Changes in Planar Figures

1. (Instructor only) a. Corresponding lengths: New length = scale factor × old length. Or, the ratios \[
\frac{\text{new length}}{\text{old length}}
\] are all equal to the same value, the scale factor.

   Corresponding angles: New angle size = old angle size.

   b. The ratio of the perimeters is equal to the scale factor. The length of each side of the original is multiplied by the scale factor, so the sum of the new lengths is also (using the distributive property, symbolically).

2. a. Not similar. Even though the corresponding angles are all the same size, the given sides would involve different scale factors, \[2 \text{ and } 1\frac{5}{7}\] : \[12 = 2 \times 6, \text{ but } 13 = 1\frac{5}{7} \times 7\] .

   b. Not similar. Even though all the sides are related by the same scale factor (\[\frac{2}{3}\] or \[1\frac{1}{2}\], depending on your view), the corresponding angles are not the same size.

3-6. Usually you can tell by looking at your result whether you have carried out the size transformation correctly; sides should be parallel to their images.

7. b. A side and its image appear to be parallel. (This gives a visual check for accuracy.)

   c. The lengths of the sides of the smaller triangle are about 2 cm, 3 cm, and 3.5 cm, and of the larger triangle, about 3.5 cm, 5.2 or 5.3 cm, and about 6.1 cm. These suggest a
scale factor of about 1.75. The angle sizes are about 88°, 34°, and 58° in each triangle. Again, a side and its image appear to be parallel.

8. a. One pair of parallel sides have lengths 30.5 m; the other pair 12.2 m. The angle sizes are 115° and 65°.
b. (b, c Instructor only) $3\frac{1}{2}$ m, $1\frac{1}{2}$ m (students may have decimals); 115°, 65°
c. Each image is a parallelogram.

9. a. $a = 50°, b = 60°, x = 7.2$ km, $y = 5\frac{1}{2}$ or $5\frac{3}{8}$ km if measurements are perfect.
b. The primes (A', B', C') tell what correspond. $p = 40°, r = 60°, q = s = 80°, x = 6$ cm, $y = 2\frac{2}{3}$ cm. (bc Instructor only)
c. $p = 12°, q = 120°, y = 2.5 \times 129 = 322.5$ yd, $x = 36$ yd
d. $r = 75°, s = 75°$. So $x$ and $28$ correspond, as do $y$ and $28.3$, and $24$ and $30$. The $24$ and $30$ give the scale factor. $x = 22\frac{2}{5} = 22.4$ cm, $y = 35\frac{3}{8} = 35.375$ cm

10. Yes, in both cases. If the center is on the figure, there are segments that overlap with rays, so it is visually different from the other cases.

11. The image is the same size but in a different place.

12. A change in only the scale factor will result in a change in the size and location of the image.

13. The image will be just like the original (although it may have moved if a rotation, and so on, is involved). The two will be congruent.

14. a. Just one point!
b. After multiplying by $-2$, measure in the opposite direction through the center.

15. Two are incorrect. Instructor: Sample corrections--a. 200% should be 100%, or 60 should be 90; b. 150% should be 50%, or 12 should be 20 (or 8 should be 4.8).

16. a. 1.6, or 160%
b. 7.5 cm and 22.5 cm; 125% (the original is the other 100%)

17. a. The new segment should be 3 copies of the original.
b. The new segment should be 2.5 copies of the original.
c. The new segment should be 4 copies of the original.
(def Instructor only) d. The new region should have 9 squares.
e. The new region should have 4.5 squares.
f. The new region should have 12 squares.

18. a. The first and third express the same relationship. (Your example?)
b. The first and third express the same relationship. (Your example?)
(Instructor only) c. The first and third express the same relationship, and the second and fourth express the same relationship (but different from the first relationship).
19. a. 56 cm  
b. 80 cm  
c. 18 cm  
d. 42 cm  
e. 54 cm  (e-h Instructor only)  
f. 30 cm  
g. 150 cm  
h. 210 cm  

20. One way would be to use the ruler method. Another way would be to draw a 60° angle, mark off on its sides length 4 times the lengths of the given 60° angle, and then draw parallels to finish the shape.  

21. In (a)-(c), angle sizes are the same as in the original, and each length in the original is multiplied by the scale factor to get the length in the image.  

d. Say $P$ dollars are invested at an interest rate of 5% per year. After one year, the original $P$ dollars would become $1.05P$ dollars, as though the original is scaled up by a factor of 1.05.  

22. Map scales are usually expressed as ratios or equations, so this one might be written 28 mi:1 in. or 1 in.:28 mi or 1 inch = 28 miles. Many times the units are the same, so the scale here could be written 1,774,080:1 without any need to mention units. (There are 5280 feet in a mile and 12 inches in a foot.)  

23. 150 km. **Hint:** You will have to measure the distances on the maps.  

24. A scale of 1 cm = 100 years allows all of the likely dates to be shown, up to 2000. With that scale and starting with year 0 at 0 cm, the Magna Carta would be 12.1 cm from the start; Columbus, 14.9 cm; Declaration of Independence, 17.8 cm; French Revolution, 17.9 cm; Civil War 18.6 cm; Wright brothers 19.0 cm; Depression 19.3; WW II 19.4 cm; atomic bomb 19.5; TV 19.5+ cm; computers 19.7 cm; and so on.  

25. a. Here it is reasonable to have 1 cm = 30 million years. Then, starting with the Cambrian at 0 cm, Carboniferous would be at about 10.7 cm, Triassic at about 13.3 cm, Cretaceous at about 17.8 cm, Oligocene at 19 cm, Pleistocene at essentially the 20 cm mark.  

b. About 66.7 cm!  

26. The original and the final image are similar; the scale factor is not 7, however.  
(Instructor: 12)  

27. b. About 2, because the dimensions of the smaller squares are roughly doubled to get the larger squares.
c. Yes, by…. (Instructor: Use smaller squares than those in the given grid.)

28. a. Yes. All the angles in each triangle have 60°, so corresponding angles are the same size. And, even though for these triangles we need not check, every ratio of the lengths of corresponding sides is 7:12 (or 12:7).

b. Yes. Again, the pairs of 60° angles assure that the triangles will be similar. How would you determine the scale factor for the chosen triangles?

c. No. One triangle might have angles 90°, 45°, 45°; the other 90°, 30°, 60°.

d. Yes. Every pair of corresponding angles has 90°, and because all of the sides of each square have the same length, the equal ratios for similarity will be assured.

e. No, even though the corresponding angles all have the same size. Give an example to show that the ratios of corresponding sides will not necessarily all be equal.

f. No. Sketch a couple, avoiding regular hexagons.

g. Yes. Corresponding angles will have the same sizes, and because all the sides of each n-gon have the same length, the ratios of lengths of corresponding sides will all be equal.

29. Measure your drawings. For example, the woolly mammoth is about 7 m from tip of tail to tusks' end, and about 4.2 m tall. This termite is about 0.8 cm long.

30. a. For example, \( \frac{x'}{y'} = \frac{x}{y} \) and \( \frac{x''}{y''} = \frac{x}{y} \).

b. These ratios are all equal.

c. These ratios are also all equal. In trigonometry, the ratios are related to the angle C in the right triangle and given names like “sine \( \angle C \).”

20.2 More About Similar Figures

1. (Instructor) Corresponding lengths: New length = scale factor \( \times \) old length. Or, the ratios \( \frac{\text{new length}}{\text{old length}} \) are all equal to the same value, the scale factor.

   Corresponding angles: New angle size = old angle size.

   Corresponding areas: New area = (scale factor)\(^2\) \( \times \) old area.

   The ratio of the perimeters is equal to the scale factor, because...

   The ratio of the areas is equal to the scale factor, squared.

2. A size change like that in the section shows the relative sizes.

3. The triangles in one pair are not similar. (Instructor: Pair c)

4. a. \( x = 22.5 \text{ km}; \ y = 16 \text{ km} \). The triangles are similar because there are two pairs of corresponding angles that have the same sizes. (Instructor only)
b. A “tricky” part is seeing the correspondence after you have established similarity because the two triangles share an angle, giving a second pair of the same size along with the 114˚ angles. It may be helpful to redraw the triangles separately, with one triangle re-oriented and the shared angles marked. \( k = 5.6 \) cm, and \( n = 5.4 - 5.1 = 0.3 \) cm

\[ k = 5.6 \text{ cm}, \quad n = 5.4 - 5.1 = 0.3 \text{ cm} \]

c. The right triangles also have same-sized angles where the lines cross, so they are similar. \( p \approx 6.4 \) mi; \( q \approx 6.4 \) mi. (Instructor only: The given 10.2 is rounded)

d. 30 ft

e. Part (a), \((1.5)^2 = 2.25\); part (c), \((1.6)^2 = 2.56\)

\[ (1.5)^2 = 2.25; \quad (1.6)^2 = 2.56 \]

f. Treat the pond like the tree in part (d). From some location a convenient distance from one side of the pond (and perpendicular to the line segment showing the distance across the pond), sight to see the other side of the pond, and make a convenient right triangle like the small one in part (d).

5. a. \( 2 \frac{1}{4} \)

b. A rectangle, because size changes keep angles the same size.

c. 4 cm by \( 6 \frac{2}{3} \) cm

d. \( 26 \frac{2}{3} \) cm\(^2\) and 135 cm\(^2\). The second is \((2 \frac{1}{4})^2\) times the first.

e. Each could have been answered differently, because the 4 cm might have referred to the longer side and hence correspond to the 15 cm side rather than the 9 cm one.

6. \( x = 45 \) cm, using a scale factor of 1.5 (or \( \frac{2}{3} \), depending on your viewpoint)

7. a. Ask about the sizes of two angles in the remote triangle, and check them against the angle sizes in your triangle.

b. Unless the quadrilaterals are special, you would have to find out about the sizes of 3 of the angles (why not 4?) and the lengths of the four sides, for each quadrilateral. If the angles can be paired so that their sizes are equal, then you would have to check the ratios of the lengths of the paired sides.

### 20.3 Size Changes in Space Figures

1. This exercise is important, because it requires that you summarize some of the major relationships for similar polyhedra, and they reinforce the relationships for similar 2D shapes.

   Corresponding lengths: New length = scale factor \times\ old length. Or, the ratios \( \frac{\text{new length}}{\text{old length}} \) are all equal to the same value, the scale factor.
Corresponding angles: New angle size = old angle size.

Corresponding areas: New area = (scale factor)$^2 \times$ old area.

Corresponding volumes: New volume = (scale factor)$^3 \times$ old volume.

2. a. 1.5
   b. 1.5
3. Shapes b and d are similar, with scale factor 2 (or $\frac{1}{2}$, depending on your view of the original); shapes b and e are similar, with scale factor $1\frac{1}{2}$ (or $\frac{2}{3}$); and shapes d and e are similar, with scale factor $\frac{3}{4}$ (or $1\frac{1}{3}$).

4. Shapes b, d, f, and g are congruent (and therefore similar), and shapes b, d, f, and g are similar to shape a. h. $2^3 = 8$, because the shapes are similar.

5. b. Because $36 = 12 \times 3$, $84 = 12 \times 7$, and $96 = 12 \times 8$ and all the corresponding angles are right angles, the prism in b is similar to the given one. Alternatively, we could say that each of the ratios $36:3$, $84:7$, and $96:8$ is equal to $12:1$ or $12$ (and the corresponding angles are all equal), so the prisms are similar.

There are three other definite “Yes” and one “Well, probably, taking rounding into account.” (Instructor: Besides b, the ones similar to the 3-7-8 prism are in c (s.f. = 2.9), e (s.f. = 2.54, because 1 inch = 2.54 cm), f (s.f. = 1$\frac{2}{3}$ probably), and i (s.f = 0.5).)

j. Yes. The scale factor is about 4.14.

6. a. 12 feet $\times$ 12 inches per foot = (scale factor) $\times$ 8 inches, ..., scale factor = 18. You might also get $\frac{1}{18}$ by using a different viewpoint as to the original figure, and the latter way is probably more common: “A $\frac{1}{18}$ scale model.”
   (Instructor) b. 2 $\frac{2}{3}$ inches, which is more meaningful to people than $\frac{6}{35}$ feet.

7. (Instructor) a. The matching angles will be the same size; several edges in P will be 2 cm or 3 cm or 4 cm, and their matches in Q will be about 5.3 cm, 8 cm, and about 10.7 cm.
   b. The scale factor is 8:3, or $2\frac{2}{3}$.
   c. $\frac{3}{8}$

8. With a scale factor of $\frac{1}{30}$, the height in the scale model should be $\frac{1}{30} \times 40$ inches, so the son is correct (the roommate probably subtracted 30 inches). The other dimensions should be $1\frac{1}{3}$ inches for the length and $\frac{2}{3}$ inch for the width.

9. a. 8
   b. 27
   c. Yes, because...
   d. From the theory, one is $(1\frac{1}{2})^2 = 2\frac{1}{2}$ times as large as the other.
e. The ratio of the volumes is $27 : 8$, but $\frac{27}{8} = \frac{3^3}{2^3} = (\frac{3}{2})^3$, the cube of the scale factor.

10. a. Yes, with scale factor... (Instructor: Scale factor = 1)
    b. Yes. The two scale factors are reciprocals, or multiplicative inverses.
    c. Yes, X and Z are similar. It is tricky to keep track of new and original, but using Z as the original and X as the final, the scale factor is (s.f. from Z to Y) × (s.f. from Y to X).
       Try a specific example to see how the product of the two scale factors enters in.

11. c. The ratio of the two should be the square of your scale factor.
    d. The ratio of the two should be the cube of your scale factor.

12. (Instructor) a. $20r$ cm, $20r$ cm, and $25r$ cm
    b. $r$
    c. Original: $2800$ cm$^2$, new one $2800r^2$ cm$^2$, so the ratio is $r^2$.
    d. Original: $1000$ cm$^3$, new one $1000r^3$ cm$^3$, so the ratio is $r^3$.

13. Twice as large could refer to lengths, or to areas, or to volumes, and the three of these are related by different powers of 2.

14. a. $8.8$ cm by $13.2$ cm by $22$ cm
    b. $3$ cm by $4.5$ cm by $7.5$ cm
       (c-f Instructor only)
    c. $\frac{16}{3}$ cm by $\frac{24}{7}$ cm by $\frac{40}{7}$ cm, or approximately $2.3$ cm by $3.4$ cm by $5.7$ cm
    d. $14\frac{2}{3}$ cm by $22$ cm by $36\frac{2}{3}$ cm
    e. $4$ cm by $6$ cm by $10$ cm
    f. $8.8$ cm by $13.2$ cm by $22$ cm

15. The new surface area is $803.5$ cm$^2$ (the surface area of the original is $62$ cm$^2$).
    The new volume is $1399.68$ cm$^3$ (the volume of the original is $30$ cm$^3$); the new dimensions are $7.2$ cm by $10.8$ cm by $18$ cm.

16. It is not 40; recall how volumes of similar figures are related. (Instructor: 1000)

17. No. If the scale factor were 0, then the image of every point would be the center.

18. a. Yes. All the angles are $90^\circ$ and every ratio of the lengths of edges is 5:8.
    b. Yes. All the angles are $90^\circ$ and every ratio of the lengths of edges is $m:n$.
    c. No. Their angles need not be equal, nor do the ratios of lengths have to be the same.
    d. No. Give an example.
    e. No. Although the ratios of the lengths of edges are all equal to 1:3, there is no assurance that the angles in the rhomboidal prism are $90^\circ$, as in the cube.

19. It is not possible to set up a correspondence so that corresponding angles are the same size. (Instructor only)
20. Use the same pattern as the given net so that the angles are the same, but multiply each segment in the net by your scale factor.

21. “Each cubic centimeter becomes a 4 cm by 4 cm by 4 cm cube in the enlargement, so each original cubic centimeter is now \(4^3\) cubic centimeters.”

22. a. 25,600 cm\(^2\)
   b. 204,800 cm\(^3\)

23. a. 3 (or \(\frac{1}{3}\)) The ratio of the volumes, 810:30, is 27, and that is the cube of the scale factor.
   b. \(3^2 = 9\) (or \((\frac{1}{3})^2 = \frac{1}{9}\))

24. (Instructor only) The new cube was actually 8 times as large in volume as the previous one, so the magician felt that the people had not followed her instruction. She should have been pleased, however, because the new cube had much more gold than she asked for.

25. (Instructor only) The nets should be nets for cubes, of course, with sides/edges for the larger one twice the lengths for the smaller one.

26. Under the assumption that your shape is similar to that of the Statue of Liberty, your answers might be about 4 ft 6 in. for the statue's nose and 3 ft for its width of mouth.

21.1 Planar Curves and Constructions

1. a. NT  b. AT  c. ST  d. AT  e. AT  
f. AT  g. AT  h. AT (Inst only)

4. a. Any line through the center will give a reflection symmetry, and any size rotation with center of rotation at the center of the circle will give a rotational symmetry.
   b. There are many reflection symmetries (all the lines of reflection will be parallel) and many rotational symmetries (all 180°, but with different centers).
   c. “Flattening out” a piece of a sphere leads to some distortion.

5. Your sketches should show the appropriate part of the circle or the circular region.
   a. \(\frac{1}{4}\)  b. \(\frac{3}{4}\)  c. \(\frac{1}{5}\)
   d. Because the whole circle has 360°, 80° would be \(\frac{80}{360}\), or \(\frac{2}{9}\), of the circumference.
   e. \(\frac{1}{5}\)  f. \(\frac{1}{8}\)  g. \(\frac{3}{8}\)  h. \(\frac{1}{6}\)

6. a. 84°  b. 42°  c. 42°  d. 42°  e. 96°  f. 180°  g. 90°  h. 90°  i. 90°

7. The angle at A is the inscribed angle, intercepting the \(d\) arc. Triangle ABC is isosceles because of the radii, so the angles at A and B are equal. The size of the central angle = 2 x the size of the inscribed angle. Or, the size of the inscribed angle is half that of the
central angle. Instructor: The sketches below suggest how to extend the argument to other cases.

8. You may check your work by measuring.
9. a. You can get the midpoint of a segment by constructing the perpendicular bisector of the segment.
   b. Instructor: The segments should pass through the same point (the center of gravity, or centroid, of the triangular region).
10. Instructor: Look for the construction marks. Parts (c) and (d) and perhaps (f) involve extending the given segment. Parts (e) and (f) are not literally illustrated in the text description.
11. Your eyes can usually tell you whether your work is all right.
12. a. Because the two distances are from the same radius, they are equal.
   b. All the points appear to lie on the perpendicular bisector of segment AB.
   c. The center will be where two perpendicular bisectors of chords intersect. The radius can then be found by measuring from the center to any point on the arc (about 4 cm here). Notice that you can use any three points of the arc to find the center of the circle.
   d. Because concentric circles share the same center, the perpendicular bisectors of chords from the different arcs will meet at the common center of the circles.
13. a. *Hint:* $45 = \text{half of } 90$
   b. *Hint:* $135 = 90 + 45$
   c-d. *Hint:* $\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$, and $\frac{1}{4}$ is half of a half. $\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$
   e. Bisect the given segment to get half of the segment. Bisect one of the half segments to obtain quarters. Bisect the quarter closest to the half mark to get eighths. Then a half segment with an eighth segment will be $\frac{5}{8}$ of the original segment.
14. a. Suppose a vandal or a malicious drunk came by. (Instructor: A curve of constant width is needed, to avoid the vandal from dropping the cover down the hole.)
   b. Yes. The same idea works with any regular polygon.
15. You may check your accuracy by measuring the central angle and comparing that to the
360° for the whole circular region.

16. **Hint:** See Learning Exercise 5.

17. Instructor: Various cycles—the lunar cycle, the cycle of the seasons, the apparent shape
of the moon,…

18. a. Usually just a visual check will do.
   
   b. Instructor: Answers will vary; check for work.

19. Using A as center, draw a circle with radius AB. Using B as center, draw a circle with
radius AB. Either of the two points in which the circles intersect can be used for C.

20. Instructor: You will recognize the (a) ASA, (b) SAS, and (c) SSS situations. Part (d)
allows you to review these, if you wish.

21. a. Yes (SAS or side-angle-side)
   
   b. Not necessarily, but they are similar
   
   c. Yes (SSS, or side-side-side)
   
   d. Yes, the third angle in GHI makes ASA (angle-side-angle) with Triangle 7.

### 21.2 Curved Surfaces

1. a. NT (Inst)  
   b. AT (Inst)  
   c. AT (but sometimes just 1)
   
   d. AT (but sometimes just 1 nontrivial one)
   
   e. AT  
   f. AT  
   g. AT

2. a. right circular cylinder  
   b. right circular cylinders
   
   c. right circular cone  
   d. right circular cone
   
   e. oblique cone  
   f. right circular cylinder

3. a. Using the line through the centers of the bases as axis, there are infinitely many
rotational symmetries. With every line along a diameter of a cross-section halfway
down as axis, there are 180° and 360° rotational symmetries as well. Any plane
perpendicular to the bases and passing through the centers of the bases will give a
reflection symmetry, along with the plane parallel to the bases and halfway down the
cylinder.
   
   b. Do you see infinitely many rotational symmetries and infinitely many reflection
symmetries? If not, reread the answer for part (b).
   
   c. Any line (or plane) through the center of the sphere will give a rotational (or
reflection) symmetry.
4. a, b Instructor only: See nets for Shapes N and O.
c. “Flattening out” a piece of a sphere leads to some distortion.
5. Instructor: a. soda or soup can, hot dog with ends cut off squarely, and so on
   b. Some ice-cream cones or drinking cups or snow-cone containers, dunce hat, and so on
   c. Smooth balls (not footballs), meatballs, jawbreakers, the Earth (roughly) and other
      planets and their moons, and so on
   d. The equator, or any circle that passes through both the North and South Poles, and so on
6. e. Not possible—every two great circles on a sphere will intersect.
7. Instructor: cone
8. a. If the layers are the same thickness, the whole cylinder would hold \(22\frac{2}{3}\) cups.
   b. 66 ml \((\text{Hint}: \text{How much will } \frac{1}{2} \text{ of the cylinder hold? } \frac{1}{4}\text{?})\)
9. a. The larger sphere has diameter with length twice the diameter of the smaller sphere.
   b. Instructor: These are not intuitive results. Because the spheres are similar, the volume
      of the larger is \(2^3 = 8\) times as large as that of the smaller, and the surface of the larger
      is \(2^2 = 4\) times as large as that of the smaller.
10. a. Circles or ellipses, if the resulting cylinder is a circular cylinder.
    b. Two of the open curves may be of different types, if the cutting plane is or is not
        parallel to some position of the generating lines.

22.1 Some Types of Rigid Motions

1. (Instructor only) Mirror image = reflection image; slide = translation; flip = reflection in
   a line; turn = rotation about a point.
2. a. B reflection, C translation, D rotation, E rotation, F translation
   b. Each is congruent to shape A, because it is the image of A for a rigid motion.
3. a. translation (unless you focus on the wheels only)
   b. rotation
   c. translation (if the slide is straight)
   d. rotation
   e. rotation
   f. rotation (either as it turns, or as the door opens)
4. 

\[ \text{shapes and measurement} \]

\[ \text{answers and hints (instructor's version)} \]

\[ \begin{array}{c}
\text{a} \\
\text{b} \\
\text{c} \\
\text{d} \\
\text{m} \\
\text{n} \\
\end{array} \]

\[ \text{each is congruent to the original, because each is the image of the original for some rigid motion.} \]

5. 

a. Translations leave the orientation unchanged.

b. Rotations leave the orientation unchanged.

c. Reflections change the orientation.

6. All make sense for 3D shapes. (Instructor only)

7. There are at least 14. Decision-making likely involves all of the rigid motions.

(Instructor: Following is an attempt to be systematic, building all 2-triangle ones and 3-triangle ones and then using the 3-triangle ones to build the 4-triangle ones:
22.2 Finding Images for Rigid Motions

1-2. Visual checks usually reveal whether images have been located correctly. Rotation images are the most difficult for many people.

4.
Each of the segments is 6 spaces long. (And each is directed toward the east.)
8. a. The segments are not all the same length. Points farther from the center have to “travel” farther than do points closer to the center.

b. 

```
  C
 /  \
A---------------B
 /    |
 /     |
 /      |
 /       |
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 /          |
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 /______________________________|
```

CA and CA'' have the same length, as do CB and CB'', and CD and CD''. This makes sense for a rotation because each point should stay its distance from the center, C.

c. Each of the angles is 90°. This result makes sense for a rotation, because each point “travels” through the same angle.

9. The segments can have different lengths, but each is perpendicular to the line of reflection.

10. a, d, g all give the same translation. c, e, f, and j all describe another translation.

11. c and d will have the same effect on every point of the plane; turning 180° one clock direction will end a shape up in exactly the same place as will 180° in the other clock direction.

22.3 A Closer Look at Rigid Motions

1. Again, your eyes usually tell you whether the result is all right, keeping in mind that most people have more trouble with rotations.

2. a. The vector should end at the figure to the left. Which shape is the original does matter, because the direction of the vector is opposite if the other figure is chosen as the original.

b. For a reflection, either shape as the original gives the same line of reflection.

c. The clock direction of the angle depends on which shape is the original.

d. Notice that this reflection is one of the lines of symmetry.

3. a. reflection in line m (see the following diagram; construction marks not shown)

b. translation with vector v

c. reflection in line n

d. rotation, center D, angle about 120° counterclockwise

e. rotation, center E, angle 180°
4. a. Every point on line \( k \) is a fixed point.
   b. There are no fixed points.
   c-d. There is one fixed point, namely...
   e. Every point is a fixed point, for the 360° rotation.

5. a-b.

6. a. Line \( n \) itself
   b. Ray \( BC \); ray \( BA \)

22.4 Composition of Rigid Motions

1. Lines \( k \) and \( m \) are perpendicular to the direction of the translation (and therefore are parallel) and are 1.5 cm apart. Note that 1.5 cm is one-half the 3 cm.

2. a. translation, 1.2 cm east   b. translation, 3 cm southwest
c. translation.... (Instructor: The vector has length twice the distance between the parallels, and direction perpendicular to the parallels, from first line of reflection toward second.)

d. rotation.... (Instructor: The center of the rotation is the intersection of the lines of reflection; the angle size is twice the angle made by the intersecting lines, from the first line toward the second.)

e. rotation... (Instructor: Center P, angle size $x + y$ degrees, clockwise.)

f. rotation... (Instructor: Center P, angle size $|a - b|$ degrees, with the direction depending on which of $a$ and $b$ is greater.)

g. rotation.... (Instructor: Without more theory, the center can be located only visually or by using the original and final shapes, but the angle of the rotation will be 110° clockwise. As in Exercise 5(b) of Section 22.3, with different centers and sum of angles algebraically = 0, the composition is a translation.)

h. reflection in a different line through T (Instructor: More theory would enable us to describe the location of the line of reflection exactly, without using the original and final shapes.)

i. glide-reflection (Instructor: More theory would enable us to describe the location of the glide-reflection exactly, without using the original and final shapes.)

3. a. A glide-reflection changes the orientation of a figure.

b. Translations and rotations do not change the orientation of a figure, but reflections and glide-reflections do.

c. The image will be congruent to the original, because each of the motions making up a glide-reflection gives a figure congruent to its original.

4. a. a translation or a rotation (Each of the originals keeps the orientation the same, so their composition will also.)

b. a translation or a rotation

c. a translation or a rotation

d. a single reflection or a glide-reflection

e. even: translation or rotation; odd: reflection or glide-reflection

f. a reflection or a glide-reflection, because...

5. b. The line of reflection appears to pass through the midpoints of the segments.

6. a. In general, yes. For example, …(try two reflections)
b. Order does not matter for the two motions defining a glide-reflection. (That is why
the definition demands that the line of reflection and the vector of the translation be
parallel.)

7. a. rotation, because the orientations are the same, and a translation won’t work. (You
may also have just “eyeballed” it.)
b. reflection, because...
c. translation, because...
d. reflection, because...
e. glide-reflection, because...

8. Q'R'S'T' by a glide-reflection; Q"R"S"T" by a rotation.

9. a. Your two lines of reflection should be perpendicular to the line of the vector, and half
the vector’s length apart.
b. Your two lines of reflection should intersect at the center of the rotation; the angle
they make will be half the size of the angle of the rotation.
c. Perhaps surprisingly, there are many possibilities in each case.

11. Any rigid motion is one of these: reflection, translation, rotation, or glide-reflection.
Each of translation/rotation can be achieved by the composition of two reflections
(Exercises 9-10), and a glide-reflection can be achieved by the composition of three
reflections (two for the translation, plus the separate reflection of the glide-reflection).

12. a. 1 or 3 reflections      b. 2 reflections

13. a. glide-reflection      b. translation      c. glide-reflection
d. rotation
 e. rotation or translation, depending on the type of tuner (dial or lever)
f. rotation or translation, depending on the type of thermostat

14. No (Instructor: For our purposes, the translation has a non-zero vector, so reflections and
glide-reflections are different.)

15. The rigid “motion” that leaves every point in the same place; this is legitimate, even
though no net motion results. (Instructor: You may have introduced the idea of the
identity transformation.)

16. The composition will be a single reflection. The line of reflection will be parallel to the
other lines. If the original lines are, from the left, $x$ units and $y$ units apart, with $x > y$ and
the composition is done starting at the right, then the composition line will be $y$ units to
the right of the third line of reflection (or $x$ units to the left of the first line of reflection).
22.5 Transformations and Earlier Topics

1. a. There are many translation symmetries, as well as many reflection symmetries. Describe several of the translation symmetries.
b. There are many translation symmetries, as well as many rotational, reflection, and glide-reflection symmetries.

2. Only one of the statements is true in each part. (Instructor: The first statement in each part is the true one.)

3. a. (two, yes, two) reflection symmetries, (two) rotational symmetries (counting the 360° rotation)
b. (only one) reflection symmetry
c. reflection, rotational, translation, glide-reflection
d. (one) reflection symmetry

4. a. There are several different translation symmetries, as well as reflection and rotational symmetries. There are also glide-reflection symmetries.

5. Hint: What is the size of the third angle in each triangle? (Instructor: a and d are similar, as are b and c.)

6. a. Yes, just use the reverse of the rigid motion.
b. Yes, use a 360° rotation.
c. Yes, use the composition of the two rigid motions.

7. The image will be congruent to the original shape.

8. a. Yes,…
b. Yes, use a scale factor of 1.
c. Yes,…

9. (Instructor: For example, two equilateral triangles with sides of different lengths. The triangles will be similar.)

10. a. Each of the triangles is similar to each of the others. Any will tessellate the plane.
b. The shape has 6 reflection symmetries and 6 rotational symmetries. It will tessellate the plane (assume the inner white pieces are filled with some color).
c. The shape has 6 rotational symmetries, but no reflection symmetries.
d. It is likely to have translation symmetries, and perhaps other sorts. It will tessellate the plane.

11. (Instructor: Two rectangles with dimensions not related by the same ratios.)

23.1 Key Ideas of Measurement

1. a. length    b. volume, although weight could conceivably be used
c. temperature    d. weight    e. area
f. speed  g. time  h. area
i. area  j. Your ideas, besides student achievement?

(Instructor only e-i)

2. (Other correct answers are possible.)
   a. yards, feet, inches, centimeters (!), millimeters (!!!)
   b. milliliters, or ounces if weight is used
   c. degrees Fahrenheit or degrees Celsius
   d. grams or ounces
   e. square centimeters or square inches
   f. meters per second or yards per second
   g. minutes
   h. square meters or square yards
   i. square meters or square yards
   j. students’ scores on tests or students’ later success rate or…

3. a. temperature, height
   b. height, weight, mathematics achievement
   c. running speed, walking speed, respiration rate—all of which involve measuring other quantities as well

4. a. barometric pressure  b. population density  c. area
   Instructor:  d. 24 (sometimes 25) sheets of paper  e. area (used for land)  f. length (about 18 inches)  g. luminous intensity  h. weight  i. electrical charge
   j. viewership  k. intensity of sound  l. hotness of peppers  m. general fitness  n. area (640 acres, or 1 square mile)

5. a. The stretchiness of a rubber band might lead to an inconsistent unit, in repeating it.
   b. The ice cube would melt, leaving no permanent unit for later use.  (a b Instructor only)
   c. “One person's junk is another person's treasure,” so the unit would have different values depending on who was using it.
   d. You have probably read about perception experiments in which people's judgments of relative temperature were vastly influenced when one finger was in hot water, as compared to when the finger was in cold water.
e. The actual amount in a pinch likely varies from person to person.

f. Different people, or even the same person at a more thirsty time, likely would have differently sized sips.

6. a. Real-life length measurements cannot be 100% exact.
   b. The fish could weigh as little as 122 \( \frac{1}{2} \) pounds and up to 123 \( \frac{1}{2} \) pounds. (b Instructor only)

7. a. Show segments as short as 6 \( \frac{1}{2} \) units and up to 7 \( \frac{1}{2} \) units.
   b. Show segments as short as 7 \( \frac{1}{4} \) units and up to 7 \( \frac{3}{4} \) units in length. (b Instructor only)

8. a. Most likely not, because a measurement of 151.0 to the nearest half-pound covers a range of possible values (as light as 150.75 or as heavy as up to 151.25).
   b. 161.75 pounds \( \leq \) a possible weight < 162.25 pounds
   c. 128.25 pounds \( \leq \) a possible weight < 128.75 pounds
   d. Both 116.0 pounds and 223.5 pounds could be as much as 0.25 pounds off. (a, d Instructor only)

9. a. \( x > 2000 \) because it would take more than 2000 raisins to weigh 2000 pounds.
   b. \( x < 2000 \) because a raisin’s weight is smaller than a pound.
   c. \( x > 2000 \) because…
   d. \( x < 2000 \) because…
   e. \( x < 2000 \) because… (e-h Instructor only)
   f. \( x > 2000 \) because…
   g. \( x < 2000 \) because…
   h. \( x > 2000 \) because…

10. a. The ap unit is larger than the ag, because it takes fewer aps to equal 150 ags.
    b. The ba unit is larger than the bo, because… (\textit{Hint: Recall that } 1.31 \times 10^3 = 1310 .) 
    c. The cin unit is larger than the con, because… (e Instructor only)

11. Weigh yourself holding the puppy, and then...

12. a. 8 \( \frac{1}{2} \) units, by counting repetitions of the unit; “cutting” the region into individual and part units and totaling them
    b. 8 units, by…\textit{Hint for exact reasoning:}
    c. 6 \( \frac{1}{2} \) units, by…
d. 4 units, by...

e. $2 \frac{1}{2}$ new units, because...; 2 new units because...; $1 \frac{5}{8}$ new units because...

13. Probably $8 \times $2.89 = $23.12, because some leftover pieces from a section would be too short to be useful elsewhere.

14. a. $17 \frac{1}{8}$ inches
   c. $5 \frac{5}{8} + 7 \frac{13}{16} + 3 \frac{1}{4} = 16 \frac{11}{16}$ inches
   d. Just under $5 \frac{2}{8} + 7 \frac{15}{16} + 3 \frac{3}{4} = 17 \frac{9}{16}$ inches

   e. From $16 \frac{11}{16}$ inches to just under $17 \frac{9}{16}$ inches (whereas the part (a) length would incorrectly imply the more narrow range of possible lengths between $17 \frac{1}{16}$ inches and $17 \frac{3}{16}$ inches).

15. The individual scores given could have been as low as 81.5, 72.5, and 86.5, giving a sum of 240.5. The 240.5 would have been shown as 241. (Instructor only)

16. a. 0.763 b. 270 c. 1.9 d. 46.2
   e. 620 f. 0.108 g. 87 h. 2.9
   i. 1.55 (e-i Instructor only)

17. a. 1300 ms
   b. $143 \times 10^{-9} \text{s} = 1.43 \times 10^{-7} \text{s}$
   c. 0.500 s or 0.5 s
   d. $2.5 \times 10^{-3} \text{s}$

18. a. 0.1532 kg
   b. 3400 mg
   c. 2170 g
   d. 0.056 g (c, d Instructor only)

19. a. 80 km
   b. 15 cm

20. 1

21. Instructor: Even within one country at that time, from community to community there were likely to be different systems of measurement. Hence, the idea of a single system would involve an almost incredible changing of established practices.

22. Correct: only parts (a) and (d). *Hints:* The calculations are done as though the relationships are base-ten relationships, but 1 hour = 60 minutes, not 100; 1 pound = 16 ounces, not 10; 1 gallon = 4 quarts, not 10.
23. Rather than what appears to be a random collection of terms, as in the given list, the metric system uses prefixes with a basic term for all of its units of a particular type.

24. a. 16
   b. In order, starting with mouthful: \(\frac{1}{64}, \frac{1}{32}, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024\)
   c. Units are related by powers of the same number. 2 is used instead of 10.

25. About \(0.9144018 - 0.9144 = 0.0000018\) meter shorter

### 23.2 Length and Angle Size

1. Recall that measurements are not exact, so answers close to those given likely reflect a correct approach. (Production processes also can change lengths slightly.)
   a. \(\approx 31.2\) cm  
   b. \(\approx 10.5\) cm  
   c. \(\approx 10.6\) cm  
   d. A: about 20 cm; B: 2 faces each—about 14.4 cm, 18.2 cm, 15.4 cm

2. Measurements are approximate.

3. Any segment with length \(\geq 6.25\) cm but \(< 6.5\) cm would fit the conditions.

4. Try doing these mentally, if you did them with paper-pencil.
   a. \(3 \frac{3}{4}\) inches  
   b. \(3 \frac{1}{8}\) inches  
   c. \(3 \frac{9}{16}\) inches

5. a-d Instructor only.  a. 10 cm  
   b. 7 cm  
   c. 0.6 cm  
   d. 2.3 cm
   h. Just under 2.5 cm, if you do not have a quarter handy.
   i. Almost 28 cm
   j. About 99 cm, or about 1 m

7. Remember that the sides have to fit together to make a polygon: 1 cm, 1 cm, 1 cm, and 21 cm would be impossible. The complete answers are not given here.
   a. Try to be systematic, possibly this way: 6-6-6-6; 5-6-6-7; 4-6-6-8; 5-6-6-7; 4-6-6-8; 3-6-6-9; 2-6-6-10; 1-6-6-11.
   b. 1, 11, 1, 11 cm; 2, 10, 2, 10 cm; 3, 9, 3, 9 cm; 4, 8, 4, 8 cm; 5, 7, 5, 7 cm; 6, 6, 6, 6 cm.
   Notice how much easier the problem is with a rectangle, because the opposite sides must be the same length.

8. Experiment, gathering and organizing the data from several simpler, specific values for \(n\) (for example, \(n = 1, n = 2, n = 3\), etc.), and look for a pattern. Instructor: For \(n\) square regions, the maximum perimeter will be \(2n + 2\), possibly justifying the general result by “seeing” the \(2n + 2\) (n across the top and bottom, plus the 2 ends) for a long strip, or by thinking \(23 + 2(n - 2)\), or \(1 + 2n + 1\).
9. a. By measuring the distance of the line segment perpendicular to the line(s) from the point to the line or from one parallel line to the other.
   b. No, so long as the distance is measured along a perpendicular.
10. The distance between the top and bottom parallel sides.
11. Ancient Greeks used this idea: The perimeter (and area) of the circle should be between the perimeters (and areas) of the outer and inner polygons.
12. a. 1200 m       b. 530 cm       c. 6.2 cm       d. 32.5 cm       (c, d Instructor only)
    e. 3.35 cm + 4.25 cm = 7.6 cm. Remember that “3.4 cm” implies the measurement is accurate to the nearest tenth-centimeter.
13. 4 and a little more, using a 2 cm unit; somewhat under 9, using a 1 cm unit; just under 18, using a 0.5 cm unit. The 0.5 cm fit very closely, so an estimate of just under 9 cm should be a good estimate.
14. a. About 90°       b. About 60°      c. About 145°
    d. right angle a (or close to it), acute angle b, obtuse angle c. No straight angle is shown; are you clear about what a straight angle would look like?
16. a. 270°       b. 35/60, or 7/12, of 360° is 210°       c. 30°
    d. 174°       e. 360°       f. 540°       g. 618°       h. 864°
    (16 cdef Instructor only)
17. 21 600 minutes, so 1 296 000 seconds. One second would then be \( \frac{1}{1296000} \) of a full turn.
18. a. 17° 43' 37"
    b. 46° 28' 43"
    c. 54° 40'
    (18c Instructor only)
19. \textit{Hint}: 360 ÷ 15 =...
20. 25 000 ÷ 360 = ...
21. Remember to check your drawings by measuring with a protractor.
22. a. \( y = 180 - x \), so then \( ? = 180 - y = 180 - (180 - x) = 180 - 180 + x = x. \)
    (22b Instructor only)
23. a. $c, z$; $a, x$; $d, y$

b. No

c. Corresponding angles of parallel lines appear to be the same size.

d. The other pair of alternate interior angles is $c, x$. If lines are parallel, the alternate interior angles are the same size.

24. (From vertical angles and the $180^\circ$ for a straight angle) $e, g$: $128^\circ$; $f$: $52^\circ$; $h, j$: $75^\circ$; $i$: $105^\circ$. (From the parallel lines and vertical angles) $k, r$: $75^\circ$; $p, q$: $105^\circ$; $a, d$: $128^\circ$; $b, c$: $52^\circ$. (Using inscribed angles and alternate interior angles of parallels) $s = 40^\circ$, $t = 40^\circ$, $v = 40^\circ$. Angle $u$ intercepts the rest of the half-circle, or $100^\circ$, so $u = 50^\circ$. (24 Instructor only)

25. a. Hint: What are the sizes of the other angles at vertex A, in terms of $z$ and $y$?

b. $69^\circ$

c. $49^\circ$

d. $60^\circ 38'$

e. $80^\circ$ and $80^\circ$; $20^\circ$ and $140^\circ$

f. $90^\circ$

26. $a=y$; $b=z$; $c=x$ $d=z$; $e=w$; $f=y$; $g=x$

27. a. Because the $n$ angles total ... and they are all equal in size, each is ...

b. Your table should show these results: $3---60^\circ$; $4---90^\circ$; $5---108^\circ$; $6---120^\circ$; $7---128^{\frac{2}{3}}$; $8---135^\circ$; $9---140^\circ$; $10---144^\circ$; $11---147^{\frac{3}{11}}^\circ$; $12---150^\circ$. Notice that the angle sizes are getting larger. How do you know that they will never equal $180^\circ$?

c. The sizes get larger and larger, but less than $180^\circ$. Instructor: If you have time, consider looking at a graph and/or the angle size expression, $\frac{(n-2)180}{n} = (1-\frac{2}{n})180$.

d. Hint: $(n - 2)180 = 4500$ Instructor: $n = 27$

e. 16

28. a-b. Focus on what must happen at a given vertex, and apply Learning Exercise 27(b).

29. a. If it were $360^\circ$, it would be flat. (If it were more than $360^\circ$, it would not be convex.)
b. What polygons give (same-sized) angles that will total less than 360° at a vertex? See Learning Exercise 27(b) and be sure to see whether there is more than one possibility for a given polygon.

c. *Hint:* See Learning Exercise 27(b) again.

30. The 1080 refers to the number of degrees in the turn the athlete can make when she/he is in air. So a 1080 refers to 3 full turns. Excellent athletes can do 720s and 900s as well.

31. $65°35' \quad (Hint: \quad 120.25° = 120° + \text{how many minutes?})$

32. Look at 90° arcs on circles of quite different sizes.

34. a. $120°, 90°, 72°, \ldots$

b. *Hint:* See part (a).

35. a. $120°$  
b. $55°$  
c. $p + q$ degrees

36. The conventional way is to measure the angle formed by the two perpendiculars to the edge of the dihedral angle, at the same point on the edge, and with one in each plane making the dihedral angle.


Someone may notice $7 \times 19 = 133, 180 - 133 = 47,$ and ask about doubling that.

38. a. Additive comparison: The 100-yd dash is 8.56 m (or 9.36 yd) shorter than the 100-m dash.

Multiplicative comparison: $100 \text{yd} : 100 \text{m} = 0.9144 : 1$, or $100 \text{m} : 100 \text{yd} = 1 : 0.9144 = 1.0936 : 1$

The mile run is about 109 meters (or about 120 yd) longer than the 1500-m run. The ratio $1760 \text{yd} : 1500 \text{m}$ would most likely be given with the units the same. $1760 \text{yd} : 1640 \text{yd} = \ldots$, or $1609 : 1500$.

39. a. 7040  
b. $\frac{3}{4}$  
c. $\frac{1}{4}$  
d. 7920  
e. 432  
f. $2\frac{1}{5}$

(39 Instructor only)

40. Instructor only a-e: a. college degree, temperature degree, extent (“to a degree”),…

b. areas around a house,…  
c. gas meter, parking meter,…

d. usual or ordinary  
e. left angle

41. b. Although you can make definite assertions about the angles, you can say only that the ratios of lengths of corresponding sides will be equal; you do not know the value of that ratio.
42. a. 400
   b. 0.009° (This is not a temperature, even though you may know that the Celsius temperature scale was once called the centigrade scale.)
43. 117 cm
44. a. The endpoints give a square. Each angle intercepts half the circle and so has size 90°. Rotate 90° to see that the sides have the same length.
   b. A regular hexagon. Joining the points to the center of the circle shows 6 connected equilateral triangles, so each side of the hexagon has the same length, and the angle at each vertex is 120°.
   c. Octagon: Bisect the central angles in part (a); 12-gon: Bisect the central angles in part (b).

### 24.1 Area and Surface Area

1. a. 3 \( \frac{1}{6} \) hexagonal regions, 6 \( \frac{1}{3} \) trapezoidal regions; 9 \( \frac{1}{2} \) rhombus regions
   b. 3 \( \frac{1}{3} \) hexagonal regions; 6 \( \frac{2}{3} \) trapezoidal regions; 10 rhombus regions
2. a. 24 rhombus regions, because...
   b. 31 \( \frac{1}{2} \) rhombus regions, because... (b Instructor only)
3. Here are possibilities, if you cannot think of any: nail of little finger for square centimeter, VW Beetle door for square meter, floor of an ordinary classroom (10 m by 10 m, or roughly 30\( ^{+} \) ft. by 30\( ^{+} \) ft.)
4. Instructor: a. 6 dm\(^2\)  b. Varies, perhaps 4 cm\(^2\)
   c. Varies, perhaps 1 are, more likely in square meters
   d. Varies, square meters likely unit
   e. Varies, might be in dm\(^2\) or m\(^2\), depending on size
5. a. $20 > x$, because if the measurements are equal, then it will take more of the smaller units to give an area equal to $x$ of the larger units (or, ...it will take fewer of the larger units to make an area equal to 20 of the smaller units).
b. $20 > x$, because…
c. $20 > x$, because… (c, d, g Instructor only)
d. $x > 20$, because…
e. $y > x$, because…
f. $x > y$, because…
g. $\frac{24}{7}$ or $3 \frac{3}{7}$ unit Is make 1 II. $\frac{7}{24}$ IIs makes 1 I.
h. $\frac{8}{3}$ IIs make 1 III. $\frac{3}{8}$ IIIs make 1 II.
i. $\frac{37}{24}$, or $\frac{37}{24}$, or $1 \frac{13}{24}$ blobs make 1 glob. $\frac{24}{37}$ glob makes 1 blob.

6. a. 64 of Beth's unit, $10 \frac{2}{3}$ of Cai's unit
b. $2 \frac{4}{5}$ of Al's unit, $1 \frac{7}{12}$ of Cai's unit
c. $9 \frac{1}{2}$ of Al's unit, 38 of Beth's unit (c Instructor only)

7. a. Your sketch should confirm, for example, that there are 144 square inches in 1 square foot. Hence, 1 square inch is $\frac{1}{144}$ of a square inch.
b. ...9 sq. ft in a square yard; $\frac{1}{9}$ sq. yd makes a square foot

c. Hint: There are 1760 yards in 1 mile.
Instructor: $1760 \times 1760 = 3,097,600$ square yards in a square mile

8. 640 acres in a square mile, perhaps from a dictionary or from $\frac{5280 \times 5280}{43,560}$

9. a. 230 b. 0.45 c. 0.196 d. 400 (9cd Instructor only)

10. 10 m

11. a. $\approx 26$ units b. $\approx 13$ units c. $\approx 19$ units d. $\approx 24$ units (11cd Instructor only)

12. a. 19 units b. Hint: “Surround” the triangle with a rectangle. 10.5 units
c. “Cut off” a slice from the left end and put it on the right end. 25 units
d. 26 units e. 26 units f. 195 units (12g-j Instructor only)
g. 11 units h. 8 units i. $5 \frac{1}{2}$ units j. 15 units

13. a. Find the areas of one of the 8 lateral faces, multiply that by 8, and add twice the area of one of the bases. What might be an efficient way to find the area of a base?
c. Find the area of one face and multiply that by 8.

14. a. 104 units b. 82 units (14a Instructor only)
15. Instructor: See Exercise 4 in Section 21.2.
16. No. The second one has area 16 square meters.
18. First region: around 13 cm$^2$; second region: around 9.5 cm$^2$
19. a. The different sizes are $\frac{1}{4}$ (regions I), $\frac{1}{8}$ (III, IV, V), and $\frac{1}{16}$ (regions II). Cutting out the pieces and moving them around should convince you.
   b. $11\frac{1}{4}$ area units (I), $5\frac{2}{5}$ area units (III, IV, V), and $2\frac{13}{16}$ area units (II).
   c. $1\frac{1}{2}$ area units (I), $\frac{7}{12}$ area unit (III, IV, V), and $\frac{7}{24}$ area unit (II).
20. $\approx 20$ units. “Cutting” the region into pieces, getting an approximate measurement of each piece, and then totaling is one method.
21. a. Ollie is incorrect, because… (Hint: See Learning Exercise 7b.)
   b. 270 dm$^2$, 27000 cm$^2$, 2.7 m$^2$…
22. The original 64 squares now seem to fill 65 squares! This strange result would violate the cutting-up key idea. The secret is that the apparent diagonal of the rectangle is really a very thin but long region, with area 1 square region.
23. Your sketch should show that the original square centimeter now occupies 25 square centimeters.
24. a. $24010$ cm$^2$, because areas are related by the square of the scale factor.
   \[ SA(\text{larger}) = 7^2 \cdot 490 \]
   b. 10 cm$^2$, because... (24b Instructor only)
   c. Not necessarily, because the pyramids are not similar ($4:6 \neq 6:8$).
25. The scale factor is either 2 or $\frac{1}{2}$, depending on which region is the original, because the ratio of the two areas, 4, say, is the square of the scale factor.
26. a. Add the lengths of the three sides of the rectangle, and half the circumference of the circle.
   b. Subtract half the area of the circular region from the area of the rectangular region.
27. Your work should show that figures with the same perimeter may not have the same area.
   Instructor: Consider 1 by 3 and 2 by 2 rectangles.
28. Each is incorrect, usually because of the inappropriateness of the unit for the quantity.

24.2 Volume
1. The units given here are possible, but others can be defended. The important thing is that the units be of the correct kind and not too large or small. Units like km or cm$^2$ for the area of a lake are either not appropriate (km is for length) or sensible (cm$^2$ is too small).
   a. volume, km$^3$ (possibly m$^3$) b. area, km$^2$ c. length, m or dam
   d. length, km e. area, m$^2$ or dam$^2$ or km$^2$
f. volume, m$^3$  

i. length, m  

l. area, m$^2$ or dam$^2$  

b. mL, or cm$^3$  

d. L or kL or m$^3$  

c. mm$^3$, possibly cm$^3$  

e. dm$^3$ or cm$^3$  

h. volume, m$^3$  

j. volume, m$^3$  

k. volume, km$^3$  

m. length, m  

n. area  

(1def Instructor only)

2. If you are stuck, here are some ideas: a portion of your little finger (1 cm$^3$, 1 mL), some plastic soda containers (1 dm$^3$, 1 L), a box for a large washing machine (1 m$^3$)

3. a. cm$^3$  

b. mL, or cm$^3$  

c. mm$^3$, possibly cm$^3$  

d. L or kL or m$^3$  

e. dm$^3$ or cm$^3$  

f. L (3 Instructor only)

4. a. About 250 mL or 250 cm$^3$  

b. About a liter (slightly less)  

c. About $\frac{1}{3}$ liter  

d. Perhaps about 300 m$^3$  

5. The usual piece of paper is about 28 cm by 21.5 cm, allowing only 2 dm across and 2 dm down. So the net would not fit. Instructor: Note that the area is 602 cm, but any arrangement of the squares in a net would not fit on the paper.

6. a. $x > y$  

b. $x < y$ (6ab Instructor only)

7. These are important for classroom work. For part (a), to see the 1000 cubic centimeters in a cubic decimeter, show the 10 cubic centimeters in 1 row, then the 100 in the 10 rows in one layer, then the 1000 in the 10 layers.

   b. 1000  
   
   c. 1000 (1 L = 1 dm$^3$)  

   d. 1 000 000  

8. a. 0.001  

b. 0.000001  

c. 0.001 (Think metric.)  

(8abc Instructor only)

9. a. 10 times, 100 times, 1000 times  

b. 100 times, 10 000 times, 1 000 000 times (9b Instructor only)

10. a. 3280  

b. 32.8  

c. 0.2257  

d. 2.257 (10cd Instructor only)

11. a. Your drawings should suggest that 1 cubic foot = 1728 cubic inches; 1 cubic yard = 27 cubic feet; OR 1 cubic inch = $\frac{1}{1728}$ cubic feet; 1 cubic foot = $\frac{1}{27}$ cubic yard.

   b. 7.48 (often rounded to 7.5)  
   
   c. About 325 830  

12. $\frac{7}{12}$ cup. The key idea is the thinking of the measurement of a whole amount in terms of measurements of its parts. (Instructor)

13. Key to the second question: Your answer to, "Does anything include idealizations that exist just in the mind?"

14. a. Surface area = 150 square regions; volume = 108 cubic regions  

b. S. A. = 238 sq. regions; volume = 222 cubic regions (Instructor)

   c. S. A. = 160 sq. regions; volume = 105 cu. regions
d. 108 mL, 222 mL, 105 mL, respectively (1 mL = 1 cm³)

e. No (1 L = 1000 mL or 1000 cm³)

15. 1 gram, 1 gram

16. Most have surface area 18 square regions. (Instructor: A 2 by 2 arrangement of cubes has a surface area = 16 square regions.)

17. a. Count by rows. 42 cubic regions; 40 cubic regions
   b. 42 square regions; 40 square regions (Instructor: As a preliminary to the \( V = Bh \) formula later, point out that the answers in parts (a) and (b) are numerically the same.)
   c. The first (30 stories, each 42 cubic regions = 1260 cubic regions vs. 31 stories, each 40 cubic regions = 1240 cubic regions)

18. Six cubic regions in each layer, 5 layers = 30 cubic regions. Some additional partial cubic regions would be needed to fill in the rest—estimate 2 per layer. 40 cubic regions. Key ideas: measurements are approximate; thinking of a region in pieces, getting the measurement of each piece, and then adding those measurements to get the measurement of the whole thing.

19. a. \( \frac{13}{2} \)  
   b. \( \frac{7}{3} \)  
   c. \( \frac{1}{2} \)  
   d. \( \frac{10}{3} \)  
   e. 6  
   f. 8, 16  
   g. \( \frac{7}{8} \)  
   h. \( \frac{9}{3} \)  
   i. \( \frac{3}{4} \)  
   j. \( \frac{2}{3} \)  
   k. 48  
   l. \( \frac{7}{16} \)  

(19defg Instructor only)

20. a. Congruent shapes have the same measurements, so S.A. = 32 square regions and volume = 12 cubic regions for the first shape, and S.A. = 34 square regions, volume = 9 cubic regions for the second shape.
   b. Surface area of the hidden, larger version of the first shape = 800 square regions; volume = 1500 cubes. For the hidden, larger shape for the second one, S.A. = 850 square regions and volume = 1125 cubic regions. (second shape Instructor only)

21. a. 20 800 cm²; 192 000 cm³  
   b. 2925 cm²; 10 125 cm³ (21b Instructor only)  
   c. 53 248 cm²; 786 432 cm³  
   d. 6292 cm²; 31 944 cm³

25.1 Circumference, Area and Surface Area Formulas

2. a. \( 5\pi \) cm, or 15.7 cm 
   b. \( 320\pi \) km, or 1004.8 km, or 1005 km. (So this is about \( 1005 \times 1000 \times 100 \) cm, or about \( 1.005 \times 10^8 \) cm.)
c. The radius is about 3 cm, so $C = 6\pi$ cm, or 18.8 cm, or about 19 cm.

d. $10\ cm + 2\pi\ cm$, or 16.3 cm

3. 3981 miles; 7962 miles (often referred to as 4000 miles and 8000 miles). *Note:* The earth is not a perfect sphere but is often treated as such.

4. a. $\frac{22}{7} \approx 3.1428571$, so...
   b. $0.038\ cm$

c. The radius of the orbit is 200 miles more than the earth’s radius, or about 4181 miles.
   Then the difference from the calculations with the two approximations of $\pi$ is 13.3 miles, which might be important if there are people on the satellite. (How does the answer differ if you use 4000 miles as the radius? Are you surprised?)

5. c (The others are only approximations.) (Instructor only)

6. 31.8 cm, or 32 cm

7. Instructor only: Perhaps surprisingly, all but humans taller than about 6’ 4” could walk under the rope, and the same result would apply on larger or many smaller spheres. The problem involves comparing the two radii: $\frac{C + 40}{2\pi} - \frac{C}{2\pi} = \frac{40}{2\pi} \approx 6.37'$

8. a. “Let me draw a parallelogram on graph paper. Check it out.”
   b. “Remember how we put two triangles together to make a parallelogram. Then to find the area of one triangle we had to take half the area of the parallelogram.”

9. a. $12.16\ cm^2$
   b. $76.5\ cm^2$
   c. $54\ cm^2$
   d. $40\ cm^2$ (9d,e Instructor only)
   e. $20\ cm^2$ each, even though their areas may look different

10. Create a parallelogram region, and see how it is related to the triangular region.

11. Make sure that your heights make right angles with the sides. In the right triangle, $d$ is the altitude for side $e$, and vice versa. Two of the heights for the obtuse triangle are outside the triangle, and the sides $i$ and $g$ must be extended to see the heights (but one would use just $g$ and $i$ as the bases, not the extensions).

12. It should not matter, but actual measurements may be off enough to make the calculated areas seem to be different. You should get the same area no matter which side you choose as base.

13. Each way should give an area of 24 square regions.
14. Each of the triangles has the same area, $7\frac{1}{2}$ units, even though the triangles may appear to the eye to have different areas. Because parallel lines are the same distance apart no matter where you measure it, the heights of all the triangles are the same (and the base is the same for each of them).

15. So long as the height is measured between the lines the bases are on, it does not matter where so long as it is measured along a perpendicular.

16. (Instructor: If the reproducing process has altered the dimensions, these answers may not be correct. Instructor only 16b)
   a. $8.99 \text{ cm}^2$
   b. $12.18 \text{ cm}^2$
   c. Cut each of the bases into 4 equal segments, and join them. (It will not be visually obvious; check with the formula.)

17. a. 62 square inches  
   b. The area is multiplied by 9: $558 \text{ in}^2$. (17b Instructor only)

18. (18i parts bdfh Instructor only)
   i. shape a: $15 \text{ cm}^2$  
      shape b: $30 \text{ cm}^2$  
      shape c: $29.75 \text{ cm}^2$
      shape d: $8.4 \text{ cm}^2$  
      shape e: $4.5\pi + 9$, or about $23.14 \text{ cm}^2$
      shape f: also $23.14 \text{ cm}^2$  
      shape g: $12 + 9\pi$, or about $40.27 \text{ cm}^2$
      shape h: $28 + 4\pi$, or about $40.57 \text{ cm}^2$
   ii. shape i. Area = $mj + \frac{1}{2}(m + n)(k – j)$
       shape j. Area = $\frac{1}{2}(pq + rs)$
       shape k. Hint: Sketch in the rest of two opposite semicircles; then the radius = $\frac{1}{4}t$.
       Area = Area (two whole circular regions) + square region = $= \frac{\pi + 2}{8}t^2$.
       shape l. Area = $v(w - \frac{r}{2}) + \frac{1}{2}\pi(v^2)^2 = \ldots$
   iii. For shapes a, b, and i, the “slant” sides cannot be predicted at this time. (If you remember the Pythagorean theorem and if the shape is symmetric, as it appears to be, you can make progress.)

   Perimeters:  
   c—$22.3 \text{ cm}$;  
   d—$12 \text{ cm}$;  
   e and f—$6\pi \text{ cm}$;  
   g—$10 + 3\pi \text{ cm}$;  
   h—$18 + 2\pi \text{ cm}$

   Shape j: Perimeter = $p + q + r + s$
   Shape k: Perimeter = $\pi t$
   Shape l: Perimeter = $v + 2(w - \frac{r}{2}) + \pi \frac{r}{2} = \ldots$

19. a. What are the areas of the two pizzas? (Assume that they have the same thickness.) The larger pizza is cheaper by the square inch. If you were surprised by the answer, put the
smaller pizza on top of the larger one and look at a slice of each with the same central angle.

(Instructor: Prices per square inch: smaller—13.9¢; larger—12.3¢. Surprised students may be focusing on just the difference in the diameters.)

b. No, the larger one costs about 15.1¢ per square inch, but the smaller costs only about 13.7¢ per square inch.

20. a. 12.96 cm$^2$
   
   b. 1 cm$^2$ for the first grid; 0.25 cm$^2$ for the second
   
   c. The second grid should give a better estimate, unless you are a fantastic estimator!

21. a. $48\pi$ or 150.8 cm$^2$
   
   b. $13.842\pi$ or 43.43 cm$^2$
   
   c. $80\pi$ or 251.33 cm$^2$

   d. Look at the fraction $\frac{\text{size of central angle}}{360}$; that will tell you what part of the whole circular region the sector is. Multiply that fraction by the area of the whole circle, $\pi r^2$. (21c,d Instructor only)

   e. This time, consider $\frac{\text{actual length of arc}}{\text{whole circumference}}$, or $\frac{\text{actual length of arc}}{2\pi r}$. Multiply that fraction by the area of the whole circle, and continuing:

   f. $24 + 8\pi$ or 49.13 cm
   
   g. $20 + 16\pi$ or 70.27 cm (for part (c))

22. In the first figure, assuming that the side of the square is $s$ units long (= the diameter of the circular region), the percent uncovered = 100% – the percent covered = 100% – $\frac{(\frac{s}{2})^2}{s^2} = 100\% - \frac{s}{4} \approx 21.5\%$. In the second figure, ...

   (Instructor: A more elegant way is to cut the second and third regions into miniature versions of the first, and see that the percent uncovered is the same in each.)

23. a. $\text{SA} \approx 200,000,000$ or $2 \times 10^8$ sq. miles.
   
   b. One way: The land area is about 30% of the total area, so find 30% of the SA from part (b): about $6 \times 10^7$ sq. miles.
   
   c. $10^8$ sq. miles (23b,c Instructor only)

24. a. Because the globes are similar, the ratio of their surface areas will be the square of the scale factor: $(\frac{16}{12})^2 = (\frac{4}{3})^2 = \frac{16}{9} \approx 1.78$. The larger globe has 1.78 times as much area as the smaller one does.

   b. smaller: $576\pi$ sq. in.; larger: $1024\pi$ sq. in. (Does this check with part (a)?)

25. a. $592 \text{ cm}^2$
b. 35,200 cm², or 3.52 m² (25b Instructor only)
c. 22 \frac{1}{3} ft², or 3216 in²
d. 2xy + 2xz + 2yz, or 2(xy + xz + yz)

26. 2880 cm²

27. The estimate could be improved by using narrower rectangles.

28. Look at the units that would result; neither the product nor the sum of four lengths would give square units.

29. a. 22.5 cm² b. 156.25% c. 56.25% (Why not 25%?)

30. Recall that the scale factor affects each length measurement. See what happens to the old length and the old width, so when you multiply the two new lengths, each involves a scale factor, giving (scale factor)² times the product of the old lengths. The areas of triangular regions are multiplied by k² also, because each of the two length measurements in the formula introduces a factor of k.

31. a. See the right triangles? b. Rhombuses and squares, because...

32. If you get stuck, look at \# dots on \# dots inside, which is close to the correct number. The final result that does give the area is called Pick's formula.
   Instructor: \[ A = \frac{\# \text{ dots on}}{2} + \# \text{ dots inside} - 1. \]

33. a. See the parallelogram region? How is its area related to the area of the triangle?
   b. See the “short” rectangular region? What is its area?

34. Instructor: c. The midline is the segment joining the midpoints of the sides that are not necessarily parallel; its length is half the sum of the lengths of the two parallel sides (that is, it is the average of the two).

35. a. A triangle
   b. \[ A(\text{triangle as special "trapezoid"}) = \frac{1}{2}(0 + b)h = \frac{1}{2}bh \]

36. b. “Cut” the general triangle into two right triangles.

37. \[ A(\text{sphere}) = \text{area of the cylinder's curved surface (without the two bases of the cylinder)} \]

25.2 Volume Formulas

1. In which ones are all the layers identical in volume? Instructor: c only

2. a. 840 (ac Instructor only) b. 105, if the 2 cm cubes can be cut
c. 548 cm²
3. Measurements can vary somewhat, and the production process may have changed measurements used in calculating the answers. If your answer is in the neighborhood of the listed answers, you were likely correct in your reasoning.

   a. For \( B = \) about 3.1 cm\(^2\), 1000 such pieces would need about 1860 cm\(^3\) of plastic.
   b. With \( B = 2.9 \) cm\(^2\) (or thereabouts—don’t forget the holes), the volume of plastic needed for 1000 such pieces would be 1740 cm\(^3\).
   c. With \( B = \) about 9.3 cm\(^2\), 1000 pieces would need 5580 cm\(^3\) of plastic.
   d. The key idea about the measurement of the whole equaling the sum of the measurements of its parts

4. a. The same.
   b. The top-bottom way gives a radius of \( \frac{11}{2} \) and a volume about 81.8 cubic inches, whereas the side-side way gives radius of \( \frac{8.5}{\pi} \) and a volume about 63.2 cubic inches. So the top-bottom way gives 18.6 cubic inches more, or is about 130% as large as the side-side.
   c. You can find another example of shapes having the same surface area but different volumes, but it takes some work. Hint: Examine the surface area of a 1 by 1 by 5 right rectangular prism and find a shape made with 6 cubes having the same surface area. (Instructor: The 1 \( \times \) 1 \( \times \) 5 right rectangular prism and a right prism made of 2 layers of 3 cubes in an L-shape have the same surface area, 22 units, but different volumes, 5 and 6 units, resp.)

5. (5abc Instructor only) a. 336 cm\(^3\)  
   b. 112 cm\(^3\)  
   c. 292 cm\(^2\)

6. Again, measurements can vary somewhat and the production process may alter the original dimensions, so if your answers are somewhat close to the ones here, you were probably thinking all right. (6b Instructor only)

   a. 13.4 cm\(^3\)  
   b. about 4.5 cm\(^3\) (one-third of the part (a) value)

7. (Instructor only) 2 592 276.5 m\(^3\)

8. a. 8” ball, volume = \( \frac{256\pi}{3} \approx 268.1 \) in\(^3\);
   
   16” ball, volume = \( \frac{2048\pi}{3} = 2144.7 \) in\(^3\)
   b. The areas are 64\(\pi \) in\(^2\) and 256\(\pi \) in\(^2\), so the costs of the plastic are about $0.39 and $1.56 (don’t forget to change the square inches into square feet).
   c. Because the spheres are similar, the ratio of the volumes is \( \left( \frac{1}{2} \right)^3 \) or 1:8, and the ratio of the areas is \( \left( \frac{1}{2} \right)^2 \) or 1:4.

9. a. \( \pi 8^2 \cdot 12 + \frac{1}{2} \pi 8^2 \cdot 3 = 2613.8 \) ft\(^3\)
   b. Because 2150.42 cu. in. ÷ 1728 cu. in. per cu. ft, 1 bushel \( \approx 1.24 \) cu. ft. So the silo will hold about 2613.8 ÷ 1.24, or 2108 bushels. Notice that the calculations would be simpler in the metric system.
10. a. \( S.A. = 4\pi m^2; \ V = \frac{4\pi}{3} m^3 \)
   b. \( V \approx 270,000,000,000 \) or \( 2.7 \times 10^{11} \) cu. miles (10b Instructor only)
   c. \( \approx 3.45 \times 10^{17} \) cubic miles
   d. about 1,277,000, using the rounded answers in parts (b) and (c)
11. a. 2 to 1
    b. 2 to 3
12. a-b. No, they are not congruent, but they have the same volume.
   c. \( \sqrt[3]{10} \approx 1.24 \), so about 1 \( \frac{1}{4} \) inches
13. a. \( \frac{3}{7} \) (Compare with Learning Exercise 11(b.))
    b. \( 1 - \frac{3}{7} = 47.6\% \)
14. The sphere, by about 2.4 cm (its radius is about 6.2 cm).
15. \( r = 3 \) cm for the original sphere. \( r = 0.005 \) cm for the small spheres. The number \( x \) of small spheres can be found from \( 36\pi = x \cdot \frac{4}{3}\pi(0.005)^3 \), which gives that there are
   \( 2.16 \times 10^8 \) small spheres. These will have a total surface area of \( 2.16 \times 10^8 \cdot 4\pi(0.005)^2 \), or 21600\( \pi \) cm\(^2\). The ratio of the surface areas, all smaller spheres to original sphere, is
   \( \frac{21600\pi}{36\pi} = \frac{600}{1} \), giving considerably more surface for burning. (A similar principle holds for melting ice: Broken-up ice from a block will melt faster than the block would.)
16. a. \( \frac{4}{3}\pi \left( \frac{1}{6} \right)^2 \approx 0.016 \) cubic miles
    b. \( \frac{\text{later volume}}{\text{earlier volume}} \approx 0.027 \), so the later volume is only 2.7\% of the earlier volume, quite an error. (But even the smaller asteroid could cause immense damage.)
17. b. “Cut” shape K into rectangular and triangular prisms and pyramids. Find the volume of each of those, and add them.
18. a. \( V = 72\pi + 18\pi = 90\pi \), about 282.7 cm\(^3\);
    \( S.A. = 9\pi \) (the bottom) \( + \pi 6)8 + 18\pi \), about 235.6 cm\(^2\)
    b. (18b Instructor only) \( V = 75\pi + \frac{359\pi}{3} \approx 497.4 \) cm\(^3\); \( S.A. = 25\pi + (10\pi)3 + 50\pi = 105\pi \), about 329.9 cm\(^2\)
    c. \( V = \pi r^2(h - r) + \frac{2}{3}\pi r^3 \)
    d. \( V = kmn + \frac{1}{3}mnh \)
    e. \( V = \pi r^2h \), and \( S.A. = 2\pi r^2 + 2\pi rh \). For the surface area, did you account for both bases, and did you notice that a net for a right circular cylinder includes a rectangle?
19. The volume of a **cube** with edges \( x \) units long is \( x^3 \) cubic units.
20. a. 384 cm\(^2\)
    b. \( S.A.(\text{cube}) = 6s^2 \)
    c. 512 cm\(^2\)
    d. \( V(\text{cube}) = s^3 \)
e. As measurements, the area and volume cannot be equal of course, because area and
volume are completely different characteristics. The second question comes down to
whether $6s^2 = s^3$ has any numerical solutions, and it does: $s = 6$ and the trivial (for
here) $s = 0$.

21. Recall that the scale factor multiplies each length measurement, so in the formulas...

22. a. $V = \frac{4}{3} \pi d^3$ (The relationship between the diameter and the radius is so simple that most
people do not memorize this formula.)

   b. $V = \frac{2}{3} \pi r^3$

   c. $A = \pi d^2$

   d. $A = \frac{1}{2} \pi d^2$ (for just the curved part; add $\pi(\frac{d}{2})^2$ if you included the planar part)

23. Without more restrictions on the shape of the dog house, there is no definite answer.

24. a. $x^2 + 2xy + y^2$  c. $x^2 + xy$  d. $x^2 + 5x + 6$

25. There are no "right" answers, but it is surprising that some progress can be made, using
estimations.

26.1 The Pythagorean Theorem

1. a. $p^2 + q^2 = r^2$

   b. $z^2 + x^2 = y^2$ (1bc Instructors only)

   c. $b^2 + c^2 = a^2$ (Notice that what is important is what the variables represent.)

2. Do not have either $a$ or $b$ equal to 0, and you will have a counterexample.

3. a. perimeter = 60 cm, area = 150 cm$^2$

   b. $35 + \sqrt{175}$ cm, or 48.2 cm; $7.5\sqrt{175}$ cm$^2$, or 99.2 cm$^2$

   c. 5.6 cm; 0.84 cm$^2$  d. $10 + \sqrt{50}$ cm, or 17.1 cm; 12.5 cm$^2$

   e. 14.4 m; 5.04 m$^2$  f. 7 cm; 2.1 cm$^2$

   g. 90 in.; 270 in$^2$  h. 20 m; 15 m$^2$ (3cef Instructors only)

4. a. 14.1 cm  b. $\sqrt{2s^2}$, or $s\sqrt{2}$, cm

   c. 26 cm  d. $\sqrt{m^2 + n^2}$ cm (4c Instructors only)

5. 10 cm, $\sqrt{117} \approx 10.8$ cm, and $\sqrt{145} \approx 12$ cm are the lengths of the diagonals of the faces,
and $\sqrt{181} \approx 13.5$ cm for each of the “inside” diagonals. How do you know that the
triangle inside the prism is a right triangle? (5 Instructor only)

6. a. $\sqrt{x^2 + y^2 + z^2}$  b. No, but a 4-foot one would, just barely.

   c. $e\sqrt{3}$ units
d. The diagonal of the cube is a diameter of the sphere, so... \(d = 10\sqrt{3}\) and \(r = 5\sqrt{3} = 8.7\)  
6(d) Instructors only

7. Consider a right triangle with legs 22 ft and \(x\) ft, and hypotenuse \(x\). (Instructor: Any of the ladders will do, because only 22.7 feet are needed to reach the top.)

8. More than \$70.  (Instructor: \$70.68)

9. Volume = \(28\sqrt{42} \approx 181\) cm\(^3\); surface area = \(84 + 6\sqrt{91} + 14\sqrt{51} \approx 241.2\) cm\(^2\)

10. a. \(4.5\pi\) cm\(^2\), \(8\pi\) cm\(^2\), \(12.5\pi\) cm\(^2\). The sum of the areas on the legs equals the area on the hypotenuse, just as in the Pythagorean theorem!

b. The area of the semicircular region on the hypotenuse is equal to the sum of the areas of the semicircular regions on the legs. Because \(a^2 + b^2 = c^2\) from the right triangle, \(\frac{1}{2}\pi(a^2) + \frac{1}{2}\pi(b^2) = \frac{\pi}{8}(a^2 + b^2) = \frac{\pi}{8}c^2\), and \(\frac{\pi}{8}c^2\) is also what you get from \(\frac{1}{2}\pi(\frac{c}{2})^2\).

11. a. The height of an equilateral triangle in terms of the length \(s\) of its sides is derived in the narrative, \(h = \frac{s\sqrt{3}}{2}\).

b. 10.83 cm\(^2\), 62.35 cm\(^2\), 73.18 cm\(^2\). Once again, the sum of the areas on the legs (10.83 + 62.35) is equal to the area on the hypotenuse.

c. The justification is like that in Exercise 10 (but with the triangle area formula, of course).

12. How do you know that \(\angle PVQ\) is a right angle? If you are stuck, it may be because you need \((a+b)(a+b) = a^2 + 2ab + b^2\).

13. Parts (b), (c), and (d) give Pythagorean triples. Do you see how they are related to the triple 3, 4, 5?

e. Yes, with an algebraic justification. This fact is useful for a teacher, as in part (f).

14. a. \(\sqrt{20}\) units

b. \(QR\), 2 units; \(ST\), \(\sqrt{20}\) units; \(UV\), \(\sqrt{32}\) units; \(WX\), \(\sqrt{20}\) units; \(YZ\), 10 units

c. Find the algebraic difference in the first coordinates and in the second coordinates; then...

15. One statement in each pair is false. (Instructor: In part (a), the second statement is false, and in parts (b), (c), and (d), the first statements are false. A counterexample for the first statement in part (d) comes from, for example, \(p = 4\), \(q = 6\), and \(r = 10\).)

16. a. \(A = 270\) sq. units

b. Because \(15^2 + 36^2 = 225 + 1296 = 1521\), and because \(39^2 = 1521\) also, the measurements give a right triangle. The area, from \(A = \frac{1}{2}bh = \frac{1}{2}36 \cdot 15 = 270\), agrees with the result from part (a).
17. For the 6 rows with 6 dots in each row, 20 is not correct; there are only 19! (Which one is missing?) This problem is another example to show that patterns cannot be trusted 100%. Instructor: At the time of the $5 \times 5$, a $3 \times 4 \times 5$ right triangle is possible, so one of the hypotenuses is 5. The 5 usually thought to be new in the $6 \times 6$ is not new.

18. a. $p = 10 + 2\sqrt{10}$ units $A = 15$ square units
   b. $p = 9 + \sqrt{18} + \sqrt{13}$ units $A = 13.5$ square units
   c. $p = \sqrt{8} + \sqrt{18} + \sqrt{26}$ units $A = 6$ square units ("surround" the triangle with a rectangle, and use a key idea) (18bc Instructors only)
   d. $p = 4\sqrt{5}$ units $A = 4$

19. Suggestion: Use the Pythagorean theorem for the heights of the triangles and of the pyramid.
   a. A drawing helps to see that the altitude of each triangle is 2 cm ($1.5^2 + h^2 = 2.5^2$) and the height $x$ for the pyramid is about 1.3 cm ($1.5^2 + x^2 = 2^2$). So, $SA = 3^2 + 4\left(\frac{1}{2} \cdot 3 \cdot 2\right) = 21$ cm$^2$, and $V = \frac{1}{3} Bh = \frac{1}{3} \cdot 3^2 \cdot 1.3 = 3.9$ cm$^3$.
   b. (Complication: the triangles are not all congruent.) As in part (a), the altitudes of the base-14 triangles is $\sqrt{120}$, or about 11 m, and of the base-10 triangles, 12 m. The height of the pyramid is $\sqrt{95}$ m. So, $SA = 14 \cdot 10 + 2\left(\frac{1}{2} \cdot 14 \cdot 11\right) + 2\left(\frac{1}{2} \cdot 10 \cdot 12\right)$, or about 414 m$^2$, and $V = \frac{1}{3} Bh = \frac{1}{3} (14 \cdot 10) \cdot \sqrt{95} \approx 455$ m$^3$ (19b Instructor only)

20. a. Assume that the ground covered is a circular region, that the pile has lots of rotational symmetries, and that the pile will be shaped like a cone.
   b. $16\pi$, or about 50.3 m$^3$
   c. Filling one sandbox to the top will take 0.675 m$^3$, so the pile will fill $16\pi \div 0.675$ sandboxes, about 74.

21. a. Make a drawing. Where is a right triangle? (Instructor: $\sqrt{160} \approx 12.6$ cm)
   b. An $80^\circ$ angle cuts off what part of the whole circumference? That length is the circumference of the base of the cone; now what is the radius? The height? The volume? (Instructor: radius of base of cone $= \frac{16}{5}$ in. Height of cone $\approx 7.8$ in. $V \approx 25.8$ in$^3$)

22. One way: How are areas of similar shapes related? (Instructor: $A = 3360$ cm$^2$; second way—find sides of larger triangle, then its area.)

23. Values are in feet. P to A: $\sqrt{300} + 2 \approx 19.3$; P to B: $\sqrt{200} + 2 \approx 16.1$;
   P to C: $\sqrt{2100} + 2 \approx 47.8$; P to D: $\sqrt{1400} + 2 \approx 39.4$. Total $\approx 122.6$ ft.

24. Use a key idea of measurement and Learning Exercise 11(a). (Instructor: $\frac{1}{6}$ of area of whole circular region – area of equilateral triangle $\approx 9.1$ cm$^2$)
25. The volumes of cubes with edges equal to legs \(a\) and \(b\) and hypotenuse \(c\) of a right triangle are related by the given equation.

26. \textit{Hint:} Place one of the boards across the corner of the canal to make an isosceles right triangle.

27. Instructor: Because the large squares are the same size, \(c^2 + 4\left(\frac{1}{2}ab\right) = a^2 + b^2 + 4\left(\frac{1}{2}ab\right)\)

28. Stuck? Does the Pythagorean theorem help? (Instructor: a. 31° in each triangle, 1.8 cm in the larger, with 1.9 cm and 1.7 cm sides in the smaller one. b. Larger: 58°, vertical leg 230 cm, other leg \(x\) cm, plus \(\frac{1}{2}x\) for the corresponding leg = 600 cm, so \(x = 368.2\), hypotenuse 430.0 cm; smaller, 32° and 58°, leg 236.8 cm, hypotenuse 280.3 cm.)

29. In order: \(\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5},\ldots\)

### 26.2 Some Other Kinds of Measurements

4. a. 97.5 ms, up to 98.5 ms  
b. 149.5 m, up to 150.5 m  
c. 1.415 L, up to 1.425 L

5. If, say, your measuring container is smaller than a container of liquid, you can remove amounts with your measuring container, keeping a record of each amount and then adding these amounts when the large container is empty.

6. These might give examples: costs of lettuce, breathing rates, gasoline mileage rates, and so on.

8. Rather than what appears to be a random collection of terms, the metric system uses prefixes with a basic term for all of its units of a particular type.

10. a. Slightly under 3.3  
b. About 3.2  
c. About 41 mph  
d. About .328

11. a. Slightly under 26.1  
b. About 24.8  
c. About 22.2  
d. About 81 kg or 178 lb  
e. About 8.7 kg or 19 lb, at least

12. a. One way: Showing on the drug number line that \(\frac{1}{4}\) is \(\frac{1}{3}\) of \(\frac{3}{4}\), so \(\frac{1}{4}\) g should use \(\frac{1}{3}\) of 12 mL, or 4 mL, on the water number line.  
b. So 1 g, or 4 one-fourths, should take 4 times as much, 16 mL  
c. 1 \(\frac{1}{2}\) g  
d. Each 4 mL uses \(\frac{1}{4}\) g, so 20 mL would use \(\frac{5}{2}\), or \(1\frac{1}{2}\), g.
e. \( \frac{3}{8} \) g  
Instructor: It is more likely that decimals would be used in Learning Exercise 12, because the measurements are metric. If someone mentions this likelihood, acknowledge that they are correct, but that the authors like to keep fraction ideas fresh.

13. a. Because 100 Celsius degrees match 180 Fahrenheit degrees, the Celsius degree is larger: 1 C° = 1.8 F°
b. 20° C would be \( \frac{20}{100} \), or \( \frac{1}{5} \) of the way from 0° C to 100° C. The corresponding Fahrenheit temperature should then be \( \frac{1}{5} \) of the way from 32° F to 212° F, or \( \frac{1}{5} \) of 180, or 36°. Starting from 32° gives a temperature of 68° F.
c. Reasoning as in part (b), 40° C corresponds to a temperature \( \frac{3}{5} \) of 180, or 72°, below 32° F. 40° F (This is the only temperature at which the two scales give the same reading.)

14. a. km/s, because a kilometer is longer than a meter
b. km/s, because 1 km per sec = 3600 km per hour
c. m/s, because 1 m/s = 3600 m/h = 3.6 km/h
d. yard/s, because 1 yd/s = 3600 yd/h and there are only 1760 yards in a mile.

15. a. \( \frac{1}{25} \) gallon per mile  (the 25 miles per gallon means 25 miles per 1 gallon)
b. About 9.1 kilometers per liter

16. a. 10 m/s  
b. 36 km/h (if the runner could run that fast for an hour)
c. The runner’s rate is 10 meters for every second.
d. The runner’s rate is 36 km for every hour (if …)

17. a. 10 yd/s  
b. About 20.45 mi/h
c. Metric (Learning Exercise 16) is easier, because the calculations are easier.

18. a. Using the cost per pound as a criterion, rather than the total cost, Danyell’s was most expensive ($7.20 versus $6.99 per lb for Amy’s, $6.79 per lb for Bea’s, $6.40 per lb for Conchita’s).
b. The average for the amounts actually bought is about $6.862 ($42.72 for 6.225 lb), but the average of the four costs per pound is $6.845 ($27.38 ÷ 4).

19. a. The total cost was $130, or $32.50/blouse.
b. Your meal was roughly $10 of the total $70, so you could argue that you should pay only $10. Splitting the $70 evenly would mean you would pay $14.
c. Although $2.29 is the average of the $1.99 and $2.59, you bought more of the $2.59 bags, giving a total cost of $29.88 for the 12 bags, or an average of $2.49 per bag.

20. For example, the cost per pound, the rate of eating, the cost per day per cow.
a. 22 \( \frac{3}{4} \) ¢ per day per cow
b. 24¢ per day per cow

\[
\frac{20}{3 \times 10^6 \text{ km} / \text{s} \cdot 3600 \text{s} / \text{h} \cdot 24 \text{ h} / \text{day} \cdot 365 \text{ day} / \text{yr} \cdot 1500 \text{ yr}} \times 360^\circ = 5.07 \times 10^{-14} = 0.0000000000000507^\circ
\]

21. With 8000 miles as the diameter of Earth and a world population of 6 billion,

\[
\frac{6 \times 10^9}{.29 \times 4\pi(4000)^2} = 103 \text{ people per square mile}
\]

22. a. For example, ounces/serving size, # servings/container, calories/serving
   b. 495 calories
   c. 10 000 mg (because 1% must have 100 mg)
   d. No, there are 3 grams of each. Comparing the 5% and 10% rates does not make sense, because they are based on different values.
   e. \(6 \frac{1}{4}\) servings ÷ 4.5 servings/box, or \(1 \frac{7}{18}\) boxes

23. a. 6 14 cups powdered sugar, 6 14 cups molasses, 2 1132 (about 2 132) teaspoons each of salt and soda, 16 1332 cups flour,… 1200 square inches, or 8 13 square feet.
   b. 1 4 cup sugar, 1 4 cup molasses, 3 32 tsp. each of salt and soda, 2132 cup flour,… (What would be approximations for the latter two?).
   c. For example, 16 bars per recipe, different amounts of ingredients per recipe or per 16 bars.

24. About 55.9 miles per hour (475 miles traveled in 8.5 hours). (Why is the average larger than 55 mi/h?)

25. If the number of children were distributed evenly over all the families, each family would have 2.6 children (a sharing-equally division of the total number of children by the number of families).

26. a. 1 12 days
   b. 6 days. There are different analyses possible. Here is one, using the compound unit, painter-day: It takes 6 painter-days to paint 1 house, so it will take 120 painter-days to paint 20 houses, which will take 20 painters 6 days.
   c. 14 712 days